Published by Institute of Physics Publishing for SISSA
Received: April 13, 2007
Accepted: June 6, 2007
Published: June 19, 2007

Noncommutative solitons in a supersymmetric chiral model in 2+1 dimensions

Olaf Lechtenfeld[a] **and Alexander D. Popov**[ab]

[a] *Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*
[b] *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow Region, Russia*
E-mail: lechtenf@itp.uni-hannover.de, popov@itp.uni-hannover.de

Abstract: We consider a supersymmetric Bogomolny-type model in 2+1 dimensions originating from twistor string theory. By a gauge fixing this model is reduced to a modified U(n) chiral model with $2\mathcal{N} \leq 8$ supersymmetries in 2+1 dimensions. After a Moyal-type deformation of the model, we employ the dressing method to explicitly construct multi-soliton configurations on noncommutative $\mathbb{R}^{2,1}$ and analyze some of their properties.

Keywords: Non-Commutative Geometry, Extended Supersymmetry, Integrable Field Theories, Solitons Monopoles and Instantons.

JHEP06(2007)065

Contents

1. Introduction

In the low-energy limit string theory with D-branes gives rise to noncommutative field theory on the branes when the string propagates in a nontrivial NS-NS two-form (B-field) background [1 − 4]. In particular, if the open string has N=2 worldsheet supersymmetry, the tree-level target space dynamics is described by a noncommutative self-dual Yang-Mills (SDYM) theory in 2+2 dimensions [5]. Furthermore, open N=2 strings in a B-field background induce on the worldvolume of n coincident D2-branes a noncommutative Yang-Mills-Higgs Bogomolny-type system in 2+1 dimensions which is equivalent to a noncommutative generalization [6] of the modified U(n) chiral model known as the Ward model [7]. The topological nature of N=2 strings and the integrability of their tree-level dynamics [8] render this noncommutative sigma model integrable.[1]

Being integrable, the commutative U(n≥2) Ward model features a plethora of exact scattering and no-scattering multi-soliton and wave solutions, i.e. time-dependent stable configurations on $\mathbb{R}^2$. These are not only a rich testing ground for physical properties such as adiabatic dynamics or quantization, but also descend to more standard multi-solitons of various integrable systems in 2+0 and 1+1 dimensions, such as sine-Gordon, upon dimensional and algebraic reduction. There is a price to pay however: Nonlinear sigma models in 2+1 dimensions may be Lorentz-invariant or integrable but not both [7, 11]. In fact, Derrick's theorem prohibits the existence of stable solitons in Lorentz-invariant scalar field theories above 1+1 dimensions. A Moyal deformation, however, overcomes this hurdle, but of course replaces Lorentz invariance by a Drinfeld-twisted version. There is another gain: The deformed Ward model possesses not only deformed versions of the

[1] For discussing some other noncommutative integrable models see e.g. [9, 10] and references therein.

just-mentioned multi-solitons, but in addition allows for a whole new class of genuinely noncommutative (multi-)solitons, in particular for the U(1) group [12, 13]! Moreover, this class is related to the generic but perturbatively constructed noncommutative scalar-field solitons [14, 15] by an infinite-stiffness limit of the potential [16].

In [12, 13] and [17]–[20] families of multi-solitons as well as their reduction to solitons of the noncommutative sine-Gordon equations were described and studied. In the nonabelian case both scattering and nonscattering configurations were obtained. For static configurations the issue of their stability was analyzed [21]. The full moduli space metric for the abelian model was computed and its adiabatic two-soliton dynamics was discussed [16].

Recall that the critical $N{=}2$ string theory has a four-dimensional target space, and its open string effective field theory is self-dual Yang-Mills [8], which gets deformed noncommutatively in the presence of a B-field [5]. Conversely, the noncommutative SDYM equations are contained [19] in the equations of motion of $N{=}2$ string field theory (SFT) [22] in a B-field background. This SFT formulation is based on the $N{=}4$ topological string description [23]. It is well known that the SDYM model can be described in terms of holomorphic bundles over (an open subset of) the twistor space[2] [26] $\mathbb{C}P^3$ and the topological $N{=}4$ string theory contains twistors from the outset. The Lax pair, integrability and the solutions to the equations of motion by twistor and dressing methods were incorporated into the $N{=}2$ open SFT in [27, 28]. However, this theory reproduces only *bosonic* SDYM theory, its symmetries (see e.g. [29 – 31]) and integrability properties. It is natural to ask: What string theory can describe *supersymmetric* SDYM theory [32, 33] in four dimensions?

There are some proposals [33 – 36] for extending $N{=}2$ open string theory (and its SFT) to be space-time supersymmetric. Moreover, it was shown by Witten [37] that $\mathcal{N}{=}4$ supersymmetric SDYM theory appears in twistor string theory, which is a B-type open topological string with the supertwistor space $\mathbb{C}P^{3|4}$ as a target space.[3] Note that $\mathcal{N}{<}4$ SDYM theory forms a BPS subsector of $\mathcal{N}$-extended super Yang-Mills theory, and $\mathcal{N}{=}4$ SDYM can be considered as a truncation of the full $\mathcal{N}{=}4$ super Yang-Mills theory [37]. It is believed [43, 39] that twistor string theory is related with the previous proposals [33 – 36] for a Lorentz-invariant supersymmetric extension of $N{=}2$ (and topological $N{=}4$) string theory which also leads to the $\mathcal{N}{=}4$ SDYM model.

A dimensional reduction of the above relations between twistor strings and $\mathcal{N}{=}4$ super Yang-Mills and SDYM models was considered in [44 – 47]. The corresponding twistor string theory after this reduction is the topological B-model on the mini-supertwistor space $\mathcal{P}^{2|4}$. In [47] it was shown that the $2\mathcal{N}{=}8$ supersymmetric extension of the Bogomolny-type model in 2+1 dimensions is equivalent to an $2\mathcal{N}{=}8$ supersymmetric modified U(n) chiral model on $\mathbb{R}^{2,1}$. The subject of the current paper is an $2\mathcal{N}{\le}8$ version of the above supersymmetric Bogomolny-type Yang-Mills-Higgs model in signature $(-++)$, its relation with an $\mathcal{N}$-extended supersymmetric modified integrable U(n) chiral model (to be defined) in 2+1 dimensions and the Moyal-type noncommutative deformation of this chiral model. We go on to explicitly construct multi-soliton configurations on noncommutative $\mathbb{R}^{2,1}$ for the

[2] For reviews of twistor theory see, e.g., the books [24, 25].

[3] For other variants of twistor string models see [38 – 40]. For recent reviews providing a twistor description of super Yang-Mills theory, see [41, 42] and references therein.

JHEP06(2007)065

corresponding supersymmetric sigma model field equations. By studying the scattering properties of the constructed configurations, we prove their asymptotic factorization without scattering for large times. We also briefly discuss a D-brane interpretation of these soliton configurations from the viewpoint of twistor string theory.

2. Supersymmetric Bogomolny model in $2+1$ dimensions

2.1 $\mathcal{N}$-extended SDYM equations in $2+2$ dimensions

Space $\mathbb{R}^{2,2}$. Let us consider the four-dimensional space $\mathbb{R}^{2,2} = (\mathbb{R}^4, g)$ with the metric

$$\mathrm{d}s^2 \;=\; g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \;=\; \det(\mathrm{d}x^{\alpha\dot\alpha}) \;=\; \mathrm{d}x^{1\dot1}\mathrm{d}x^{2\dot2} - \mathrm{d}x^{2\dot1}\mathrm{d}x^{1\dot2} \tag{2.1}$$

with $(g_{\mu\nu}) = \mathrm{diag}(-1,+1,+1,-1)$, where $\mu,\nu,\ldots = 1,\ldots,4$ are space-time indices and $\alpha = 1,2$, $\dot\alpha = \dot1,\dot2$ are spinor indices. We choose the coordinates[4]

$$(x^\mu) \;=\; (x^a, \tilde t) \;=\; (t,x,y,\tilde t) \qquad \text{with} \qquad a,b,\ldots = 1,2,3\,, \tag{2.2}$$

and the signature $(-++-)$ allows us to introduce real isotropic coordinates (cf. [19, 6])

$$x^{1\dot1} = \frac12(t-y)\,, \quad x^{1\dot2} = \frac12(x+\tilde t)\,, \quad x^{2\dot1} = \frac12(x-\tilde t)\,, \quad x^{2\dot2} = \frac12(t+y)\;. \tag{2.3}$$

SDYM. Recall that the SDYM equations for a field strength tensor $F_{\mu\nu}$ on $\mathbb{R}^{2,2}$ read

$$\frac12\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = F_{\mu\nu}\,, \tag{2.4}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is a completely antisymmetric tensor on $\mathbb{R}^{2,2}$ and $\varepsilon_{1234} = 1$. In the coordinates (2.3) we have the decomposition

$$F_{\alpha\dot\alpha,\beta\dot\beta} \;=\; \partial_{\alpha\dot\alpha}A_{\beta\dot\beta} - \partial_{\beta\dot\beta}A_{\alpha\dot\alpha} + [A_{\alpha\dot\alpha}, A_{\beta\dot\beta}] \;=\; \varepsilon_{\alpha\beta}\,F_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\,F_{\alpha\beta} \tag{2.5}$$

with

$$F_{\dot\alpha\dot\beta} \;:=\; -\frac12\varepsilon^{\alpha\beta}F_{\alpha\dot\alpha,\beta\dot\beta} \qquad \text{and} \qquad F_{\alpha\beta} \;:=\; -\frac12\varepsilon^{\dot\alpha\dot\beta}F_{\alpha\dot\alpha,\beta\dot\beta}\,, \tag{2.6}$$

where $\varepsilon_{\alpha\beta}$ is antisymmetric, $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma$, and similar for $\varepsilon^{\dot\alpha\dot\beta}$, with $\varepsilon^{12} = \varepsilon^{\dot1\dot2} = 1$. The gauge potential $(A_{\alpha\dot\alpha})$ will appear in the covariant derivative

$$D_{\alpha\dot\beta} \;=\; \partial_{\alpha\dot\beta} + [A_{\alpha\dot\beta},\,\cdot\,]\;. \tag{2.7}$$

In spinor notation, (2.4) is equivalently written as

$$F_{\dot\alpha\dot\beta} \;=\; 0 \qquad \Leftrightarrow \qquad F_{\alpha\dot\alpha,\beta\dot\beta} \;=\; \varepsilon_{\dot\alpha\dot\beta}\,F_{\alpha\beta}\;. \tag{2.8}$$

Solutions $\{A_{\alpha\dot\alpha}\}$ to these equations form a subset (a BPS sector) of the solution space of Yang-Mills theory on $\mathbb{R}^{2,2}$.

[4] Our conventions are chosen to match those of [12] after reduction to the space $\mathbb{R}^{2,1}$ with coordinates (t,x,y).

JHEP06(2007)065

$\mathcal{N}$-extended SDYM in component fields. The field content of $\mathcal{N}$-extended super SDYM is[5]

$$\begin{aligned}
&\mathcal{N}=0 \quad A_{\alpha\dot\alpha} && && (2.9a)\\
&\mathcal{N}=1 \quad A_{\alpha\dot\alpha},\ \chi^i_\alpha && \text{with} \quad i=1 && (2.9b)\\
&\mathcal{N}=2 \quad A_{\alpha\dot\alpha},\ \chi^i_\alpha,\ \phi^{[ij]} && \text{with} \quad i,j=1,2 && (2.9c)\\
&\mathcal{N}=3 \quad A_{\alpha\dot\alpha},\ \chi^i_\alpha,\ \phi^{[ij]},\ \tilde\chi^{[ijk]}_{\dot\alpha} && \text{with} \quad i,j,k=1,2,3 && (2.9d)\\
&\mathcal{N}=4 \quad A_{\alpha\dot\alpha},\ \chi^i_\alpha,\ \phi^{[ij]},\ \tilde\chi^{[ijk]}_{\dot\alpha},\ G^{[ijkl]}_{\dot\alpha\dot\beta} && \text{with} \quad i,j,k,l=1,2,3,4\ . && (2.9e)
\end{aligned}$$

Here $(A_{\alpha\dot\alpha},\ \chi^i_\alpha,\ \phi^{[ij]},\ \tilde\chi^{[ijk]}_{\dot\alpha},\ G^{[ijkl]}_{\dot\alpha\dot\beta})$ are fields of helicities $(+1,+\frac12,0,-\frac12,-1)$. These fields obey the field equations of the $\mathcal{N}=4$ SDYM model, namely [33, 37]

$$F_{\dot\alpha\dot\beta} = 0\,, \tag{2.10a}$$

$$D_{\alpha\dot\alpha}\chi^{i\alpha} = 0\,, \tag{2.10b}$$

$$D_{\alpha\dot\alpha}D^{\alpha\dot\alpha}\phi^{ij} + 2\{\chi^{i\alpha},\chi^j_\alpha\} = 0\,, \tag{2.10c}$$

$$D_{\alpha\dot\alpha}\tilde\chi^{\dot\alpha[ijk]} - 6[\chi^{[i}_\alpha,\phi^{jk]}] = 0\,, \tag{2.10d}$$

$$D_\alpha^{\ \dot\gamma}G^{[ijkl]}_{\dot\gamma\dot\beta} + 12\{\chi^{[i}_\alpha,\tilde\chi^{jkl]}_{\dot\beta}\} - 18[\phi^{[ij},D_{\alpha\dot\beta}\phi^{kl]}] = 0\ . \tag{2.10e}$$

Note that the $\mathcal{N}<4$ SDYM field equations are governed by the first $\mathcal{N}+1$ equations of (2.10), where $F_{\dot\alpha\dot\beta}=0$ is counted as one equation and so on.

2.2 Superfield formulation of $\mathcal{N}$-extended SDYM

Superspace $\mathbb{R}^{4|4\mathcal{N}}$. Recall that in the space $\mathbb{R}^{2,2}=(\mathbb{R}^4,g)$ with the metric g given in (2.1) one may introduce *purely real* Majorana-Weyl spinors[6] θ^α and $\eta^{\dot\alpha}$ of helicities $+\frac12$ and $-\frac12$ as anticommuting (Grassmann-algebra) objects. Using $2\mathcal{N}$ such spinors with components $\theta^{i\alpha}$ and $\eta_i^{\dot\alpha}$ for $i=1,\dots,\mathcal{N}$, one can define the $\mathcal{N}$-extended superspace $\mathbb{R}^{4|4\mathcal{N}}$ and the $\mathcal{N}$-extended supersymmetry algebra generated by the supertranslation operators

$$P_{\alpha\dot\alpha} \ =\ \partial_{\alpha\dot\alpha}\,,\qquad Q_{i\alpha}\ =\ \partial_{i\alpha}-\eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}\qquad\text{and}\qquad Q^i_{\dot\alpha}\ =\ \partial^i_{\dot\alpha}-\theta^{i\alpha}\partial_{\alpha\dot\alpha}\,, \tag{2.11}$$

where

$$\partial_{\alpha\dot\alpha}\ :=\ \frac{\partial}{\partial x^{\alpha\dot\alpha}}\,,\qquad \partial_{i\alpha}\ :=\ \frac{\partial}{\partial\theta^{i\alpha}}\qquad\text{and}\qquad \partial^i_{\dot\alpha}\ :=\ \frac{\partial}{\partial\eta_i^{\dot\alpha}}\ . \tag{2.12}$$

The commutation relations for the generators (2.11) read

$$\{Q_{i\alpha},Q^j_{\dot\alpha}\}\ =\ -2\delta_i^jP_{\alpha\dot\alpha}\,,\qquad [P_{\alpha\dot\alpha},Q_{i\beta}]\ =\ 0\qquad\text{and}\qquad [P_{\alpha\dot\alpha},Q^i_{\dot\beta}]\ =\ 0\ . \tag{2.13}$$

To rewrite equations of motion in terms of $\mathbb{R}^{4|4\mathcal{N}}$ superfields one uses the additional operators

$$D_{i\alpha}\ =\ \partial_{i\alpha}+\eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}\qquad\text{and}\qquad D^i_{\dot\alpha}\ =\ \partial^i_{\dot\alpha}+\theta^{i\alpha}\partial_{\alpha\dot\alpha}\,, \tag{2.14}$$

[5] We use symmetrization $(\cdot)$ and antisymmetrization $[\cdot]$ of k indices with weight $\frac{1}{k!}$, e.g. $[ij]=\frac{1}{2!}(ij-ji)$.

[6] Note that in Minkowski signature the Weyl spinor θ^α is complex and $\eta_{\dot\alpha}=\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}=\overline{\theta^\alpha}$ is complex conjugate to θ^α. For the Kleinian (split) signature $2+2$, however, these spinors are real and independent of one another.

which (anti)commute with the operators (2.11) and satisfy

$$\{D_{i\alpha}, D^j_{\dot\beta}\} \;=\; 2\delta_i^j P_{\alpha\dot\beta}\,, \qquad [P_{\alpha\dot\alpha}, D_{i\beta}] \;=\; 0 \qquad \text{and} \qquad [P_{\alpha\dot\alpha}, D^j_{\dot\beta}] \;=\; 0\;. \tag{2.15}$$

Antichiral superspace $\mathbb{R}^{4|2\mathcal{N}}$. On the superspace $\mathbb{R}^{4|4\mathcal{N}}$ one may introduce tensor fields depending on bosonic and fermionic coordinates (superfields), differential forms, Lie derivatives $\mathcal{L}_X$ etc.. Furthermore, on any such superfield $\mathcal{A}$ one can impose the constraint equations $\mathcal{L}_{D_{i\alpha}}\mathcal{A} = 0$, which for a scalar superfield f reduce to the so-called antichirality conditions

$$D_{i\alpha} f \;=\; 0\;. \tag{2.16}$$

These are easily solved by using a coordinate transformation on $\mathbb{R}^{4|4\mathcal{N}}$,

$$(x^{\alpha\dot\alpha},\ \eta_i^{\dot\alpha},\ \theta^{i\alpha}) \quad\to\quad (\tilde x^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - \theta^{i\alpha}\eta_i^{\dot\alpha},\ \eta_i^{\dot\alpha},\ \theta^{i\alpha})\,, \tag{2.17}$$

under which $\partial_{\alpha\dot\alpha}, D_{i\alpha}$ and $D^i_{\dot\alpha}$ transform to the operators

$$\tilde\partial_{\alpha\dot\alpha} \;=\; \partial_{\alpha\dot\alpha}\,, \qquad \tilde D_{i\alpha} \;=\; \partial_{i\alpha} \qquad \text{and} \qquad \tilde D^i_{\dot\alpha} \;=\; \partial^i_{\dot\alpha} + 2\theta^{i\alpha}\partial_{\alpha\dot\alpha}\;. \tag{2.18}$$

Then (2.16) simply means that f is defined on a sub-superspace $\mathbb{R}^{4|2\mathcal{N}} \subset \mathbb{R}^{4|4\mathcal{N}}$ with coordinates

$$\tilde x^{\alpha\dot\alpha} \qquad \text{and} \qquad \eta_i^{\dot\alpha}\;. \tag{2.19}$$

This space is called antichiral superspace. In the following we will usually omit the tildes when working on the antichiral superspace.

$\mathcal{N}$-extended SDYM in superfields. The $\mathcal{N}$-extended SDYM equations can be rewritten in terms of superfields on the antichiral superspace $\mathbb{R}^{4|2\mathcal{N}}$ [33, 48]. Namely, for any given $0 \le \mathcal{N} \le 4$, fields of a proper multiplet from (2.9) can be combined into superfields $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ depending on $x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha} \in \mathbb{R}^{4|2\mathcal{N}}$ and giving rise to covariant derivatives

$$\nabla_{\alpha\dot\alpha} \;:=\; \partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha} \qquad \text{and} \qquad \nabla^i_{\dot\alpha} \;:=\; \partial^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}\;. \tag{2.20}$$

In such terms the $\mathcal{N}$-extended SDYM equations (2.10) read

$$[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla_{\alpha\dot\beta}, \nabla_{\beta\dot\alpha}] = 0\,, \quad [\nabla^i_{\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla^i_{\dot\beta}, \nabla_{\beta\dot\alpha}] = 0\,, \quad \{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} + \{\nabla^i_{\dot\beta}, \nabla^j_{\dot\alpha}\} = 0\,, \tag{2.21}$$

which is equivalent to

$$[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}\,, \qquad [\nabla^i_{\dot\alpha}, \nabla_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^i_\beta \qquad \text{and} \qquad \{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^{ij}\,, \tag{2.22}$$

where $\mathcal{F}^{ij}$ is antisymmetric and $\mathcal{F}_{\alpha\beta}$ is symmetric in their indices.

The above gauge potential superfields $(\mathcal{A}_{\alpha\dot\alpha},\ \mathcal{A}^i_{\dot\alpha})$ as well as the gauge strength superfields $(\mathcal{F}_{\alpha\beta},\ \mathcal{F}^i_\alpha,\ \mathcal{F}^{ij})$ contain all physical component fields of the $\mathcal{N}$-extended SDYM model. For instance, the lowest component of the triple $(\mathcal{F}_{\alpha\beta},\ \mathcal{F}^i_\alpha,\ \mathcal{F}^{ij})$ in an η-expansion is $(F_{\alpha\beta},\ \chi^i_\alpha,\ \phi^{ij})$, with zeros in case $\mathcal{N}$ is too small. By employing Bianchi identities for the gauge strength superfields, one successively obtains [48] the superfield expansions and the field equations (2.10) for all component fields.

It is instructive to extend the antichiral combination in (2.18) to potentials and covariant derivatives,

$$\begin{array}{ccccc} \tilde{D}^i_{\dot\alpha} & = & \partial^i_{\dot\alpha} & + \; 2\,\theta^{i\alpha} & \partial_{\alpha\dot\alpha} \\ + & & + & & + \\ \tilde{\mathcal{A}}^i_{\dot\alpha} & := & \mathcal{A}^i_{\dot\alpha} & + \; 2\,\theta^{i\alpha} & \mathcal{A}_{\alpha\dot\alpha} \\ \| & & \| & & \| \\ \tilde\nabla^i_{\dot\alpha} & := & \nabla^i_{\dot\alpha} & + \; 2\,\theta^{i\alpha} & \nabla_{\alpha\dot\alpha} \end{array} \tag{2.23}$$

where $\nabla_{\alpha\dot\alpha}$, $\nabla^i_{\dot\alpha}$ and $\tilde{D}^i_{\dot\alpha}$ are given by (2.20) and (2.18), while $\mathcal{A}^i_{\dot\alpha}$ and $\mathcal{A}_{\alpha\dot\alpha}$ depend on $x^{\alpha\dot\alpha}$ and $\eta^{\dot\alpha}_i$ only. With the antichiral covariant derivatives, one may condense (2.21) or (2.22) into the single set

$$\{\tilde\nabla^i_{\dot\alpha}, \tilde\nabla^j_{\dot\beta}\} \;+\; \{\tilde\nabla^i_{\dot\beta}, \tilde\nabla^j_{\dot\alpha}\} \;=\; 0 \qquad \Leftrightarrow \qquad \{\tilde\nabla^i_{\dot\alpha}, \tilde\nabla^j_{\dot\beta}\} \;=\; \varepsilon_{\dot\alpha\dot\beta}\,\tilde{\mathcal{F}}^{ij}\,, \tag{2.24}$$

with $\tilde{\mathcal{F}}^{ij} = \mathcal{F}^{ij} + 4\,\theta^{[i\alpha}\mathcal{F}^{j]}_{\alpha} + 4\,\theta^{i\alpha}\theta^{j\beta}\mathcal{F}_{\alpha\beta}$. The concise form (2.24) of the $\mathcal{N}$-extended SDYM equations is quite convenient, and we will use it interchangeable with (2.21).

Linear system for $\mathcal{N}$-extended SDYM. It is well known that the superfield SDYM equations (2.21) can be seen as the compatibility conditions for the linear system of differential equations

$$\zeta^{\dot\alpha}(\partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha})\,\psi \;=\; 0 \qquad \text{and} \qquad \zeta^{\dot\alpha}(\partial^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha})\,\psi \;=\; 0\,, \tag{2.25}$$

where $(\zeta_{\dot\beta}) = \binom{1}{\zeta}$ and $\zeta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\zeta_{\dot\beta}$. The extra (spectral) parameter[7] ζ lies in the extended complex plane $\mathbb{C} \cup \infty = \mathbb{C}P^1$. Here ψ is a matrix-valued function depending not only on $x^{\alpha\dot\alpha}$ and $\eta^{\dot\alpha}_i$ but also (meromorphically) on $\zeta \in \mathbb{C}P^1$. We subject the $n{\times}n$ matrix ψ to the following reality condition:

$$\psi(x^{\alpha\dot\alpha}, \eta^{\dot\alpha}_i, \zeta)\,\left[\psi(x^{\alpha\dot\alpha}, \eta^{\dot\alpha}_i, \bar\zeta)\right]^\dagger \;=\; \mathbb{1}\,, \tag{2.26}$$

where "$\dagger$" denotes hermitian conjugation and $\bar\zeta$ is complex conjugate to ζ. This condition guarantees that all physical fields of the $\mathcal{N}$-extended SDYM model will take values in the adjoint representation of the algebra $u(n)$. In the concise form the linear system (2.25) is written as

$$\zeta^{\dot\alpha}(\nabla^i_{\dot\alpha} + 2\theta^{i\alpha}\nabla_{\alpha\dot\alpha})\,\psi \;=\; 0 \qquad \Leftrightarrow \qquad \zeta^{\dot\alpha}(\tilde{D}^i_{\dot\alpha} + \tilde{\mathcal{A}}^i_{\dot\alpha})\,\psi \;=\; 0 \qquad \Leftrightarrow \qquad \zeta^{\dot\alpha}\,\tilde\nabla^i_{\dot\alpha}\,\psi \;=\; 0\,. \tag{2.27}$$

2.3 Reduction of $\mathcal{N}$-extended SDYM to $2+1$ dimensions

The supersymmetric Bogomolny-type Yang-Mills-Higgs equations in 2+1 dimensions are obtained from the described $\mathcal{N}$-extended super SDYM equations by a dimensional reduction $\mathbb{R}^{2,2} \to \mathbb{R}^{2,1}$. In particular, for the $\mathcal{N}$=0 sector we demand the components A_μ of a gauge potential to be independent of x^4 and put $A_4 =: \varphi$. Here, φ is a Lie-algebra valued

[7] The parameter ζ is related with λ used in [45] by the formula $\zeta = \mathrm{i}\frac{1-\lambda}{1+\lambda}$ (cf. e.g. [31]).

JHEP06(2007)065

scalar field in three dimensions (the Higgs field) which enters into the Bogomolny-type equations. Similarly, for $\mathcal{N} \geq 1$ one can reduce the $\mathcal{N}$-extended SDYM equations on $\mathbb{R}^{2,2}$ by imposing the ∂_4-invariance condition on all the fields ($A_{\alpha\dot\alpha}$, χ^i_α, $\phi^{[ij]}$, $\tilde\chi^{[ijk]}_{\dot\alpha}$, $G^{[ijkl]}_{\dot\alpha\dot\beta}$) from the $\mathcal{N}$=4 supermultiplet or its truncation to $\mathcal{N}$<4 and obtain supersymmetric Bogomolny-type equations on $\mathbb{R}^{2,1}$.

Spinors in $\mathbb{R}^{2,1}$. Recall that on $\mathbb{R}^{2,2}$ both $\mathcal{N}$=4 SDYM theory and full $\mathcal{N}$=4 super Yang-Mills theory have an SL(4, $\mathbb{R}$) $\cong$ Spin(3,3) R-symmetry group [33]. A dimensional reduction to $\mathbb{R}^{2,1}$ enlarges the supersymmetry and R-symmetry to 2$\mathcal{N}$=8 and Spin(4,4), respectively, for both theories (cf. [49] for Minkowski signature). More generally, any number $\mathcal{N}$ of supersymmetries gets doubled to 2$\mathcal{N}$ in the reduction. Since dimensional reduction collapses the rotation group Spin(2,2) $\cong$ Spin(2,1)$_L\times$Spin(2,1)$_R$ of $\mathbb{R}^{2,2}$ to its diagonal subgroup Spin(2,1)$_D$ as the local rotation group of $\mathbb{R}^{2,1}$, the distinction between undotted and dotted indices disappears. We shall use undotted indices henceforth.

Coordinates and derivatives in $\mathbb{R}^{2,1}$. The above discussion implies that one can relabel the bosonic coordinates $x^{\alpha\dot\beta}$ from (2.3) by $x^{\alpha\beta}$ and split them as

$$x^{\alpha\beta} \;=\; \frac{1}{2}(x^{\alpha\beta}+x^{\beta\alpha})+\frac{1}{2}(x^{\alpha\beta}-x^{\beta\alpha}) \;=\; x^{(\alpha\beta)}+x^{[\alpha\beta]} \tag{2.28}$$

into antisymmetric and symmetric parts,

$$x^{[\alpha\beta]} \;=\; \frac{1}{2}\varepsilon^{\alpha\beta}x^4 \;=\; \frac{1}{2}\varepsilon^{\alpha\beta}\tilde t \qquad\text{and}\qquad x^{(\alpha\beta)} \;=:\; y^{\alpha\beta}\,, \tag{2.29}$$

respectively, with

$$y^{11}=x^{11}=\frac{1}{2}(t-y)\,,\qquad y^{12}=\frac{1}{2}(x^{12}+x^{21})=\frac{1}{2}x\,,\qquad y^{22}=x^{22}=\frac{1}{2}(t+y)\;. \tag{2.30}$$

We also have $\theta^{i\alpha}\mapsto\theta^{i\alpha}$ and $\eta_i^{\dot\alpha}\mapsto\eta_i^{\alpha}$ for the fermionic coordinates on $\mathbb{R}^{4|4\mathcal{N}}$ reduced to $\mathbb{R}^{3|4\mathcal{N}}$.

Bosonic coordinate derivatives reduce in 2+1 dimensions to the operators

$$\partial_{(\alpha\beta)} \;=\; \frac{1}{2}(\partial_{\alpha\beta}+\partial_{\beta\alpha}) \tag{2.31}$$

which read explicitly as

$$\partial_{(11)} \;=\; \frac{\partial}{\partial y^{11}} \;=\; \partial_t-\partial_y\,,\qquad \partial_{(12)}=\partial_{(21)} \;=\; \frac{1}{2}\frac{\partial}{\partial y^{12}} \;=\; \partial_x\,,\qquad \partial_{(22)} \;=\; \frac{\partial}{\partial y^{22}} \;=\; \partial_t+\partial_y\;. \tag{2.32}$$

We thus have

$$\frac{\partial}{\partial x^{\alpha\beta}} \;=\; \partial_{(\alpha\beta)}-\varepsilon_{\alpha\beta}\partial_4 \;=\; \partial_{(\alpha\beta)}-\varepsilon_{\alpha\beta}\partial_{\tilde t}\,, \tag{2.33}$$

where $\varepsilon_{12}=-\varepsilon_{21}=-1$, $\partial_4=\partial/\partial x^4$ and $\partial_{\tilde t}=\partial/\partial\tilde t$.

The operators $D_{i\alpha}$ and $D^i_{\dot\alpha}$ acting on $\tilde t$-independent superfields reduce to

$$D_{i\alpha} \;=\; \partial_{i\alpha}+\eta_i^\beta\partial_{(\alpha\beta)} \qquad\text{and}\qquad D^i_\alpha \;=\; \partial^i_\alpha+\theta^{i\beta}\partial_{(\alpha\beta)}\,, \tag{2.34}$$

where $\partial_{i\alpha}=\partial/\partial\theta^{i\alpha}$ and $\partial^i_\alpha=\partial/\partial\eta^\alpha_i$. Similarly, the antichiral operators $\tilde D_{i\alpha}$ and $\tilde D^i_{\dot\alpha}$ in (2.18) become

$$\hat D_{i\alpha} \;=\; \partial_{i\alpha} \qquad\text{and}\qquad \hat D^i_\alpha \;=\; \partial^i_\alpha+2\theta^{i\beta}\partial_{(\alpha\beta)}\;. \tag{2.35}$$

Supersymmetric Bogomolny-type equations in component fields. According to (2.33), the components $A_{\alpha\dot\beta}$ of a gauge potential in four dimensions split into the components $A_{(\alpha\beta)}$ of a gauge potential in three dimensions and a Higgs field $A_{[\alpha\beta]} = -\varepsilon_{\alpha\beta}\,\varphi$, i.e.

$$A_{\alpha\beta} \;=\; A_{(\alpha\beta)} + A_{[\alpha\beta]} \;=\; A_{(\alpha\beta)} - \varepsilon_{\alpha\beta}\,\varphi\;. \tag{2.36}$$

Then the covariant derivatives $D_{\alpha\dot\beta}$ reduced to three dimensions become the differential operators

$$D_{\alpha\beta} - \varepsilon_{\alpha\beta}\,\varphi \;=\; \partial_{(\alpha\beta)} + [A_{(\alpha\beta)},\;\cdot\;] - \varepsilon_{\alpha\beta}[\varphi,\;\cdot\;]\,, \tag{2.37}$$

and the Yang-Mills field strength on $\mathbb{R}^{2,1}$ decomposes as

$$F_{\alpha\beta,\,\gamma\delta} \;=\; [D_{\alpha\beta},\,D_{\gamma\delta}] \;=\; \varepsilon_{\alpha\gamma}\,f_{\beta\delta} + \varepsilon_{\beta\delta}\,f_{\alpha\gamma} \qquad \text{with} \quad f_{\alpha\beta} = f_{\beta\alpha}\;. \tag{2.38}$$

Substituting (2.36) and (2.37) into (2.10), i.e. demanding that all fields in (2.10) are independent of $x^4 = \tilde t$, we obtain the following supersymmetric Bogomolny-type equations on $\mathbb{R}^{2,1}$:

$$f_{\alpha\beta} + D_{\alpha\beta}\varphi \;=\; 0\,, \tag{2.39a}$$
$$D_{\alpha\beta}\,\chi^{i\beta} + \varepsilon_{\alpha\beta}\,[\varphi,\,\chi^{i\beta}] \;=\; 0\,, \tag{2.39b}$$
$$D_{\alpha\beta}\,D^{\alpha\beta}\phi^{ij} + 2[\varphi,\,[\varphi,\phi^{ij}]] + 2\{\chi^{i\alpha},\,\chi^j_\alpha\} \;=\; 0\,, \tag{2.39c}$$
$$D_{\alpha\beta}\,\tilde\chi^{\beta[ijk]} - \varepsilon_{\alpha\beta}\,[\varphi,\,\tilde\chi^{\beta[ijk]}] - 6[\chi^{[i}_\alpha,\,\phi^{jk]}] \;=\; 0\,, \tag{2.39d}$$
$$D_\alpha{}^\gamma G^{[ijkl]}_{\gamma\beta} + [\varphi, G^{[ijkl]}_{\alpha\beta}] + 12\{\chi^{[i}_\alpha, \tilde\chi^{jkl]}_\beta\} - 18[\phi^{[ij}, D_{\alpha\beta}\phi^{kl]}] - 18\varepsilon_{\alpha\beta}[\phi^{[ij}, [\phi^{kl]}, \varphi]] \;=\; 0\;. \tag{2.39e}$$

Supersymmetric Bogomolny-type equations in terms of superfields. Translations generated by the vector field $\partial_4 = \partial_{\tilde t}$ are isometries of superspaces $\mathbb{R}^{4|4\mathcal{N}}$ and $\mathbb{R}^{4|2\mathcal{N}}$. By taking the quotient with respect to the action of the abelian group $\mathcal{G}$ generated by ∂_4, we obtain the reduced full superspace $\mathbb{R}^{3|4\mathcal{N}} \cong \mathbb{R}^{4|4\mathcal{N}}/\mathcal{G}$ and the reduced antichiral superspace $\mathbb{R}^{3|2\mathcal{N}} \cong \mathbb{R}^{4|2\mathcal{N}}/\mathcal{G}$. In the following, we shall work on $\mathbb{R}^{3|2\mathcal{N}}$ and $\mathbb{R}^{3|2\mathcal{N}} \times \mathbb{C}P^1$, since the reduced ψ-function from (2.25) and (2.27) is defined on the latter space.

The linear system stays in the center of the superfield approach to the $\mathcal{N}$-extended SDYM equations. After imposing $\tilde t$-independence on all fields in the linear system (2.27), we arrive at the linear equations

$$\zeta^\alpha\,\hat\nabla^i_\alpha\,\psi \;\equiv\; \zeta^\alpha(\hat D^i_\alpha + \hat{\mathcal{A}}^i_\alpha)\,\psi \;=\; 0 \tag{2.40}$$

of the same form but with

$$\hat D^i_\alpha \;=\; \partial^i_\alpha + 2\theta^{i\beta}\partial_{(\alpha\beta)} \qquad \text{and} \qquad \hat{\mathcal{A}}^i_\alpha \;=\; \mathcal{A}^i_\alpha + 2\theta^{i\beta}(\mathcal{A}_{(\alpha\beta)} - \varepsilon_{\alpha\beta}\Xi)\,, \tag{2.41}$$

where $\mathcal{A}^i_\alpha$, $\mathcal{A}_{(\alpha\beta)}$ and Ξ are superfields depending on $y^{\alpha\beta}$ and η^α_i only. These linear equations expand again to the pair (cf. (2.25))

$$\zeta^\beta(\partial_{(\alpha\beta)} + \mathcal{A}_{(\alpha\beta)} - \varepsilon_{\alpha\beta}\Xi)\,\psi \;=\; 0 \qquad \text{and} \qquad \zeta^\alpha(\partial^i_\alpha + \mathcal{A}^i_\alpha)\,\psi \;=\; 0\;. \tag{2.42}$$

The compatibility conditions for the linear system (2.40) read

$$\{\hat\nabla^i_\alpha,\hat\nabla^j_\beta\}\;+\;\{\hat\nabla^i_\beta,\hat\nabla^j_\alpha\}\;=\;0\qquad\Leftrightarrow\qquad\{\hat\nabla^i_\alpha,\hat\nabla^j_\beta\}\;=\;\varepsilon_{\alpha\beta}\,\hat{\mathcal{F}}^{ij}\tag{2.43}$$

and present a condensed form of (2.39) rewritten in terms of $\mathbb{R}^{3|2\mathcal{N}}$ superfields. Similarly, these equations can also be written in more expanded forms analogously to (2.21) or using the superfield analog of (2.37). However, we will not do this since all these sets of equations are equivalent.

3. Noncommutative $\mathcal{N}$-extended U(n) chiral model in 2+1 dimensions

As has been known for some time, nonlinear sigma models in $2+1$ dimensions may be Lorentz-invariant or integrable but not both [7, 11]. We will show that the super Bogomolny-type model discussed in section 2 after a gauge fixing is equivalent to a super extension of the modified U(n) chiral model (so as to be integrable) first formulated by Ward [7]. Since integrability is compatible with noncommutative deformation (if introduced properly, see e.g. [9]–[20]) we choose from the beginning to formulate our super extension of this chiral model on Moyal-deformed $\mathbb{R}^{2,1}$ with noncommutativity parameter $\theta\geq 0$. Ordinary space-time $\mathbb{R}^{2,1}$ can always be restored by taking the commutative limit $\theta\to 0$.

Star-product formulation. Classical field theory on noncommutative spaces may be realized in a star-product formulation or in an operator formalism.[8] The first approach is closer to the commutative field theory: it is obtained by simply deforming the ordinary product of classical fields (or their components) to the noncommutative star product

$$(f\star g)(x)\;=\;f(x)\exp\{\tfrac{\mathrm{i}}{2}\,\overleftarrow{\partial_a}\,\theta^{ab}\,\overrightarrow{\partial_b}\}\,g(x)\qquad\Rightarrow\qquad x^a\star x^b-x^b\star x^a\;=\;\mathrm{i}\theta^{ab}\tag{3.1}$$

with a constant antisymmetric tensor θ^{ab}. Specializing to $\mathbb{R}^{2,1}$, we use real coordinates $(x^a)=(t,x,y)$ in which the Minkowski metric g on $\mathbb{R}^3$ reads $(g_{ab})=\mathrm{diag}(-1,+1,+1)$ with $a,b,\ldots=1,2,3$ (cf. section 2). It is straightforward to generalize the Moyal deformation (3.1) to the superspaces introduced in the previous section, allowing in particular for non-anticommuting Grassmann-odd coordinates. Deferring general superspace deformations and their consequences to future work, we here content ourselves with the simple embedding of the "bosonic" Moyal deformation into superspace, meaning that (3.1) is also valid for superfields f and g depending on Grassmann variables $\theta^{i\alpha}$ and η^α_i.

For later use we consider not only isotropic coordinates and vector fields

$$u:=\tfrac{1}{2}(t{+}y)=y^{22}\,,\qquad v:=\tfrac{1}{2}(t{-}y)=y^{11}\,,\qquad\partial_u=\partial_t+\partial_y=\partial_{(22)}\,,\qquad\partial_v=\partial_t-\partial_y=\partial_{(11)}\tag{3.2}$$

introduced in section 2, but also the complex combinations

$$z:=x+\mathrm{i}y\,,\qquad\bar z:=x-\mathrm{i}y\,,\qquad\partial_z=\tfrac{1}{2}(\partial_x-\mathrm{i}\partial_y)\,,\qquad\partial_{\bar z}=\tfrac{1}{2}(\partial_x+\mathrm{i}\partial_y)\,.\tag{3.3}$$

[8] See [50] for reviews on noncommutative field theories.

JHEP06(2007)065

Since the time coordinate t remains commutative, the only nonvanishing component of the noncommutativity tensor θ^{ab} is

$$\theta^{xy} \;=\; -\theta^{yx} \;=:\; \theta > 0 \qquad \Rightarrow \qquad \theta^{z\bar{z}} \;=\; -\theta^{\bar{z}z} \;=\; -2\mathrm{i}\,\theta\ . \tag{3.4}$$

Hence, we have

$$z \star \bar{z} \;=\; z\bar{z} + \theta \qquad \text{and} \qquad \bar{z} \star z \;=\; z\bar{z} - \theta \tag{3.5}$$

as examples of the general formula (3.1).

Operator formalism. The nonlocality of the star products renders explicit computation cumbersome. We therefore pass to the operator formalism, which trades the star product for operator-valued spatial coordinates $(\hat{x}, \hat{y})$ or their complex combinations $(\hat{z}, \hat{\bar{z}})$, subject to

$$[t, \hat{x}] \;=\; [t, \hat{y}] \;=\; 0 \quad \text{but} \quad [\hat{x}, \hat{y}] \;=\; \mathrm{i}\theta \qquad \Rightarrow \qquad [\hat{z}, \hat{\bar{z}}] \;=\; 2\,\theta\ . \tag{3.6}$$

The latter equation suggests the introduction of annihilation and creation operators,

$$a \;=\; \frac{1}{\sqrt{2\theta}}\,\hat{z} \qquad \text{and} \qquad a^\dagger \;=\; \frac{1}{\sqrt{2\theta}}\,\hat{\bar{z}} \qquad \text{with} \quad [a\,, a^\dagger] \;=\; 1\,, \tag{3.7}$$

which act on a harmonic-oscillator Fock space $\mathcal{H}$ with an orthonormal basis $\{\,|\ell\rangle,\ \ell = 0, 1, 2, \ldots\}$ such that

$$a\,|\ell\rangle \;=\; \sqrt{\ell}\,|\ell{-}1\rangle \qquad \text{and} \qquad a^\dagger\,|\ell\rangle \;=\; \sqrt{\ell{+}1}\,|\ell{+}1\rangle\ . \tag{3.8}$$

Any superfield $f(t, z, \bar{z}, \eta_i^\alpha)$ on $\mathbb{R}^{3|2\mathcal{N}}$ can be related to an operator-valued superfield $\hat{f}(t, \eta_i^\alpha) \equiv F(t, a, a^\dagger, \eta_i^\alpha)$ on $\mathbb{R}^{1|2\mathcal{N}}$ acting in $\mathcal{H}$, with the help of the Moyal-Weyl map

$$f(t, z, \bar{z}, \eta_i^\alpha) \quad \mapsto \quad \hat{f}(t, \eta_i^\alpha) \;=\; \text{Weyl-ordered } f\big(t, \sqrt{2\theta}a, \sqrt{2\theta}a^\dagger, \eta_i^\alpha\big)\ . \tag{3.9}$$

The inverse transformation recovers the ordinary superfield,

$$\hat{f}(t, \eta_i^\alpha) \;\equiv\; F(t, a, a^\dagger, \eta_i^\alpha) \quad \mapsto \quad f(t, z, \bar{z}, \eta_i^\alpha) \;=\; F_\star\Big(t, \frac{z}{\sqrt{2\theta}}, \frac{\bar{z}}{\sqrt{2\theta}}, \eta_i^\alpha\Big)\,, \tag{3.10}$$

where $F_\star$ is obtained from F by replacing ordinary with star products. Under the Moyal-Weyl map, we have

$$f \star g \quad \mapsto \quad \hat{f}\;\hat{g} \qquad \text{and} \qquad \int \mathrm{d}x\ \mathrm{d}y\ f \;=\; 2\pi\,\theta\,\mathrm{Tr}\hat{f} \;=\; 2\pi\,\theta \sum_{\ell\geq 0} \langle\ell|\hat{f}|\ell\rangle\,, \tag{3.11}$$

and the spatial derivatives are mapped into commutators,

$$\partial_z f \quad \mapsto \quad \hat{\partial}_z \hat{f} \;=\; -\frac{1}{\sqrt{2\theta}}\,[a^\dagger, \hat{f}] \qquad \text{and} \qquad \partial_{\bar{z}} f \quad \mapsto \quad \hat{\partial}_{\bar{z}} \hat{f} \;=\; \frac{1}{\sqrt{2\theta}}\,[a\,, \hat{f}]\ . \tag{3.12}$$

For notational simplicity we will from now on omit the hats over the operators except when confusion may arise.

Gauge fixing for ψ. Note that the linear system (2.40) and the compatibility conditions (2.43) are invariant under a gauge transformation

$$\psi \quad \mapsto \quad \psi' \;=\; g^{-1}\psi\,, \tag{3.13a}$$
$$\mathcal{A} \quad \mapsto \quad \mathcal{A}' \;=\; g^{-1}\mathcal{A}\,g + g^{-1}\partial\,g \qquad \text{(with appropriate indices)}\,, \tag{3.13b}$$
$$\Xi \quad \mapsto \quad \Xi' \;=\; g^{-1}\Xi\,g\,, \tag{3.13c}$$

where $g = g(x^a, \eta^\alpha_i)$ is a $\mathrm{U}(n)$-valued superfield globally defined on the deformed superspace $\mathbb{R}^{3|2\mathcal{N}}_\theta \times \mathbb{C}P^1$. Using a gauge transformation of the form (3.13), we can choose ψ such that it will satisfy the standard asymptotic conditions (see e.g. [51])

$$\psi = \Phi^{-1} \;+\; O(\zeta) \qquad\qquad \text{for} \quad \zeta \to 0\,, \tag{3.14a}$$
$$\psi = \mathbb{1} \;+\; \zeta^{-1}\Upsilon \;+\; O(\zeta^{-2}) \quad \text{for} \quad \zeta \to \infty\,, \tag{3.14b}$$

where the $\mathrm{U}(n)$-valued function Φ and $u(n)$-valued function Υ depend on x^a and η^α_i. This "unitary" gauge is compatible with the reality condition for ψ,

$$\psi(x^a, \eta^\alpha_i, \zeta)\;\left[\psi(x^a, \eta^\alpha_i, \bar\zeta)\right]^\dagger \;=\; \mathbb{1}\,, \tag{3.15}$$

obtained by reduction from (2.26).

Gauge fixing for $\hat{\mathcal{A}}^i_\alpha$. After fixing the unitary gauge (3.14) for ψ and inserting $(\zeta^\alpha) = \left(\begin{smallmatrix}\zeta\\ -1\end{smallmatrix}\right)$ in the linear system (2.40), one can easily reconstruct the superfield given in (2.41) from Φ or Υ via

$$\hat{\mathcal{A}}^i_1 \;=\; 0 \qquad \text{and} \qquad \hat{\mathcal{A}}^i_2 \;=\; \Phi^{-1}\hat{D}^i_2\Phi \;=\; \hat{D}^i_1\Upsilon \tag{3.16}$$

and thus fix a gauge for the superfields $\hat{\mathcal{A}}^i_\alpha$. The operators $\hat{D}^i_\alpha$ were defined in (2.35). One can express (3.16) in terms of $\mathcal{A}^i_\alpha$ and $\mathcal{A}_{(\alpha\beta)} - \varepsilon_{\alpha\beta}\Xi$ as

$$\mathcal{A}^i_1 \;=\; 0 \qquad \text{and} \qquad \mathcal{A}^i_2 \;=\; \Phi^{-1}\partial^i_2\Phi = p a^i_1\Upsilon\,, \tag{3.17}$$
$$\mathcal{A}_{(11)} \;=\; 0 \qquad \text{and} \qquad \mathcal{A}_{(12)} + \Xi \;=\; \Phi^{-1}\partial_{(12)}\Phi = \partial_{(11)}\Upsilon\,, \tag{3.18}$$
$$\mathcal{A}_{(21)} - \Xi \;=\; 0 \qquad \text{and} \qquad \mathcal{A}_{(22)} \;=\; \Phi^{-1}\partial_{(22)}\Phi = \partial_{(12)}\Upsilon\,. \tag{3.19}$$

Using (2.32), we can rewrite the nonzero components as

$$\mathcal{A} \;:=\; \Phi^{-1}\partial_u\Phi \;=\; \partial_x\Upsilon\,, \qquad \mathcal{B} \;:=\; \Phi^{-1}\partial_x\Phi \;=\; \partial_v\Upsilon\,, \qquad \mathcal{C}^i \;:=\; \Phi^{-1}\partial^i_2\Phi \;=\; \partial^i_1\Upsilon\,. \tag{3.20}$$

Recall that the superfields Φ and Υ depend on x^a and η^α_i.

Linear system. In the above-introduced unitary gauge the linear system (2.42) reads

$$(\zeta\partial_x - \partial_u - \mathcal{A})\,\psi \;=\; 0\,, \qquad (\zeta\partial_v - \partial_x - \mathcal{B})\,\psi \;=\; 0\,, \qquad (\zeta\partial^i_1 - \partial^i_2 - \mathcal{C}^i)\,\psi \;=\; 0\,, \tag{3.21}$$

which adds the last equation to the linear system of the Ward model [7] and generalizes it to superfields $\mathcal{A}(x^a, \eta^\alpha_j)$, $\mathcal{B}(x^a, \eta^\alpha_j)$ and $\mathcal{C}^i(x^a, \eta^\alpha_j)$. The concise form of (3.21) reads

$$\left(\zeta\,\hat{D}^i_1 - \hat{D}^i_2 - \hat{\mathcal{A}}^i_2\right)\psi \;=\; 0 \tag{3.22}$$

or, in more explicit form,

$$\left[\zeta\left(\partial^i_1 + 2\theta^{i1}\partial_v + 2\theta^{i2}\partial_x\right) \;-\; \left(\partial^i_2 + \mathcal{C}^i + 2\theta^{i1}(\partial_x + \mathcal{B}) + 2\theta^{i2}(\partial_u + \mathcal{A})\right)\right]\psi \;=\; 0\,. \tag{3.23}$$

JHEP06(2007)065

$\mathcal{N}$-extended sigma model. The compatibility conditions of this linear system are the $\mathcal{N}$-extended noncommutative sigma model equations

$$\hat{D}^i_1(\Phi^{-1}\hat{D}^j_2\,\Phi) \;+\; \hat{D}^j_1(\Phi^{-1}\hat{D}^i_2\,\Phi) \;=\; 0 \tag{3.24}$$

which in expanded form reads

$$(g^{ab}+v_c\varepsilon^{cab})\,\partial_a(\Phi^{-1}\partial_b\Phi) \;=\; 0 \quad\Leftrightarrow\quad \partial_x(\Phi^{-1}\partial_x\Phi) \;-\; \partial_v(\Phi^{-1}\partial_u\Phi) \;=\; 0\,, \tag{3.25a}$$

$$\partial^i_1(\Phi^{-1}\partial_x\Phi) \;-\; \partial_v(\Phi^{-1}\partial^i_2\Phi) \;=\; 0\,, \quad \partial^i_1(\Phi^{-1}\partial_u\Phi) \;-\; \partial_x(\Phi^{-1}\partial^i_2\Phi) \;=\; 0\,, \tag{3.25b}$$

$$\partial^i_1(\Phi^{-1}\partial^j_2\Phi) \;+\; \partial^j_1(\Phi^{-1}\partial^i_2\Phi) \;=\; 0\,. \tag{3.25c}$$

Here, the first line contains the Wess-Zumino-Witten term with a constant vector $(v_c) = (0,1,0)$ which spoils the standard Lorentz invariance but yields an integrable chiral model in 2+1 dimensions. Recall that Φ is a $\mathrm{U}(n)$-valued matrix whose elements act as operators in the Fock space $\mathcal{H}$ and depend on x^a and $2\mathcal{N}$ Grassmann variables η^α_i. As discussed in section 2, the compatibility conditions of the linear equations (3.22) (or (3.21)) are equivalent to the $\mathcal{N}$-extended Bogomolny-type equations (2.39) for the component (physical) fields. Thus, chiral model field equations (3.25) are equivalent to a gauge fixed form of equations (2.39).

Υ-formulation. Instead of Φ-parametrization of $(\mathcal{A},\mathcal{B},\mathcal{C}^i)$ given in (3.17)–(3.20) we may use the equivalent Υ-parametrization also given there. In this case, the compatibility conditions for the linear system (3.21) reduce to

$$(\partial^2_x-\partial_u\partial_v)\Upsilon \;+\; [\partial_v\Upsilon\,,\,\partial_x\Upsilon] \;=\; 0\,, \tag{3.26a}$$

$$(\partial^i_2\partial_v-\partial^i_1\partial_x)\Upsilon \;+\; [\partial^i_1\Upsilon\,,\,\partial_v\Upsilon] \;=\; 0\,, \quad (\partial^i_2\partial_x-\partial^i_1\partial_u)\Upsilon \;+\; [\partial^i_1\Upsilon\,,\,\partial_x\Upsilon] \;=\; 0\,, \tag{3.26b}$$

$$(\partial^i_2\partial^j_1+\partial^j_2\partial^i_1)\Upsilon \;+\; \{\partial^i_1\Upsilon\,,\,\partial^j_1\Upsilon\} \;=\; 0\,, \tag{3.26c}$$

which in concise form read

$$(\hat{D}^i_2\,\hat{D}^j_1+\hat{D}^j_2\,\hat{D}^i_1)\,\Upsilon \;+\; \{\hat{D}^i_1\Upsilon\,,\,\hat{D}^j_1\Upsilon\} \;=\; 0\,. \tag{3.27}$$

Recall that Υ is a $u(n)$-valued matrix whose elements act as operators in the Fock space $\mathcal{H}$ and depend on x^a and $2\mathcal{N}$ Grassmann variables η^α_i.

For $\mathcal{N}$=4, the commutative limit of (3.27) can be considered as Siegel's equation [33] reduced to 2+1 dimensions. According to Siegel, one can extract the multiplet of physical fields appearing in (2.39) from the prepotential Υ via

$$\partial^i_1\Upsilon = A^i_2\,, \quad \partial^i_1\partial^j_1\Upsilon = \phi^{ij}\,, \quad \partial^i_1\partial^j_1\partial^k_1\Upsilon = \tilde{\chi}^{[ijk]}_2\,, \quad \partial^i_1\partial^j_1\partial^k_1\partial^l_1\Upsilon = G^{[ijkl]}_{22}\,, \tag{3.28a}$$

$$\partial_{(\alpha1)}\Upsilon = A_{(\alpha2)}-\varepsilon_{\alpha2}\varphi\,, \quad \partial_{(\alpha1)}\partial^i_1\Upsilon = \chi^i_\alpha\,, \quad \partial_{(\alpha1)}\partial_{(\beta1)}\Upsilon = f_{\alpha\beta}\,, \tag{3.28b}$$

where one takes Υ and its derivatives at $\eta^2_i = 0$. The other components of the physical fields, i.e. $\tilde{\chi}^{[ijk]}_1$, $G^{[ijkl]}_{11}$, $G^{[ijkl]}_{21}$, $A_{(11)}$ and $A_{(21)}-\varphi$, vanish in this *light-cone* gauge.

JHEP06(2007)065

Supersymmetry transformations. The $4\mathcal{N}$ supercharges given in (2.11) reduce in 2+1 dimensions to the form

$$Q_{i\alpha} \;=\; \partial_{i\alpha} - \eta_i^\beta \partial_{(\alpha\beta)} \qquad \text{and} \qquad Q^i_\alpha \;=\; \partial^i_\alpha - \theta^{i\beta}\partial_{(\alpha\beta)}\ . \tag{3.29}$$

Their antichiral version, matching to $\hat{D}_{i\alpha}$ and $\hat{D}^j_\beta$ of (2.35), reads

$$\hat{Q}_{i\alpha} \;=\; \partial_{i\alpha} - 2\eta_i^\beta \partial_{(\alpha\beta)} \qquad \text{and} \qquad \hat{Q}^j_\beta \;=\; \partial^j_\beta\,, \tag{3.30}$$

so that

$$\{\hat{Q}_{i\alpha}\,,\,\hat{Q}^j_\beta\} \;=\; -2\,\delta_i^j\,\partial_{(\alpha\beta)}\ . \tag{3.31}$$

On a (scalar) $\mathbb{R}^{3|2\mathcal{N}}$ superfield Σ these supersymmetry transformations act as

$$\hat{\delta}\,\Sigma \;:=\; \varepsilon^{i\alpha}\hat{Q}_{i\alpha}\Sigma \;+\; \varepsilon^\alpha_i \hat{Q}^i_\alpha\Sigma \tag{3.32}$$

and are induced by the coordinate shifts

$$\hat{\delta}\,y^{\alpha\beta} \;=\; -2\varepsilon^{i(\alpha}\eta_i^{\beta)} \qquad \text{and} \qquad \hat{\delta}\,\eta^\alpha_i \;=\; \varepsilon^\alpha_i\,, \tag{3.33}$$

where $\varepsilon^{i\alpha}$ and ε^α_i are $4\mathcal{N}$ real Grassmann parameters. It is easy to see that our equations (3.24) and (3.27) are invariant under the supersymmetry transformations (3.32) (applied to Φ or Υ). This is simply because the operators $\hat{D}_{i\alpha}$ and $\hat{D}^j_\beta$ anticommute with the supersymmetry generators $\hat{Q}_{i\alpha}$ and $\hat{Q}^j_\beta$. Therefore, the equations of motion (3.25) of the modified $\mathcal{N}$-extended chiral model in 2+1 dimensions as well as their reductions to 2+0 and 1+1 dimensions carry $2\mathcal{N}$ supersymmetries and are genuine supersymmetric extensions of the corresponding bosonic equations. Note that this type of extension is not the standard one since the R-symmetry groups are Spin$(\mathcal{N},\mathcal{N})$ in 2+1 and Spin$(\mathcal{N},\mathcal{N})\times$ Spin$(\mathcal{N},\mathcal{N})$ in 1+1 dimensions, which differ from the compact unitary R-symmetry groups of standard sigma models. Contrary to the standard case of two-dimensional sigma models the above "noncompact" $2\mathcal{N}$ supersymmetries do not impose any constraints on the geometry of the target space, e.g. they do not demand it to be Kähler [52] or hyper-Kähler [53]. This may be of interest and deserves further study.

Action functionals. In either formulation of the $\mathcal{N}$-extended supersymmetric SDYM model on $\mathbb{R}^{2,2}$ there are difficulties with finding a proper action functional generalizing the one [54, 55] for the purely bosonic case. These difficulties persist after the reduction to 2+1 dimensions, i.e. for the equations (3.25) and (3.26) describing our supersymmetric modified U(n) chiral model. It is the price to be paid for overcoming the no-go barrier $\mathcal{N}\le 4$ and the absence of geometric target-space constraints. On a more formal level, the problem is related to the *chiral* character of (3.24) as well as (3.27), where only the operators $\hat{D}^i_\alpha$ but not $\hat{D}_{i\alpha}$ appear. Note however, that for $\mathcal{N}=4$ one *can* write an action functional in component fields producing the equations (2.39), which are equivalent to the superspace equations (3.24) when $i,j=1,\dots,4$ (see e.g. [47]).

One proposal for an action functional stems from Siegel's idea [33] for the Υ-formulation of the $\mathcal{N}$-extended SDYM equations. Namely, one sees that $\partial_2^i\Upsilon$ enters only linearly into the last two lines in (3.26). Therefore, if we introduce

$$\Upsilon_{(1)} \;:=\; \Upsilon|_{\eta_i^2=0} \tag{3.34}$$

then it must satisfy the first equation from (3.26), and the remaining equations iteratively define the dependence of Υ on η_i^2 starting from $\Upsilon_{(1)}$. Hence, all information is contained in $\Upsilon_{(1)}$, as can also be seen from (3.28). In other words, the dependence of Υ on η_i^2 is not 'dynamical'. For an action one can then take (cf. [33])

$$S \;=\; \int \mathrm{d}^3x\, \mathrm{d}^{\mathcal{N}}\eta^1\, \left\{ \Upsilon_{(1)}\partial_{(\alpha\beta)}\partial^{(\alpha\beta)}\Upsilon_{(1)} \;+\; \frac{2}{3}\,\Upsilon_{(1)}\,\varepsilon^{\alpha\beta}\partial_{(\alpha 1)}\Upsilon_{(1)}\,\partial_{(\beta 1)}\Upsilon_{(1)} \right\}\,. \tag{3.35}$$

Extremizing this functional yields the first line of (3.26) at $\eta_i^2 = 0$. Except for the Grassmann integration, this action has the same form as the purely bosonic one [55]. One may apply the same logic to the Φ-formulation where the action for the purely bosonic case is also known [54, 56].

4. $\mathcal{N}$-extended multi-soliton configurations via dressing

The existence of the linear system (3.22) (equivalent to (3.21)) encoding solutions of the $\mathcal{N}$-extended U(n) chiral model in an auxiliary matrix ψ allows for powerful methods to systematically construct explicit solutions for ψ and hence for $\Phi^\dagger = \psi|_{\zeta=0}$ and $\Upsilon = \lim_{\zeta\to\infty} \zeta\,(\psi - \mathbb{1})$. For our purposes the so-called dressing method [57, 51] proves to be the most practical [12]–[20], and so we shall use it here for our linear system, i.e. already in the $\mathcal{N}$-extended noncommutative case.

Multi-pole ansatz for ψ. The dressing method is a recursive procedure for generating a new solution from an old one. More concretely, we rewrite the linear system (3.21) in the form

$$\psi(\partial_u - \zeta\partial_x)\psi^\dagger \;=\; \mathcal{A}\,, \qquad \psi(\partial_x - \zeta\partial_v)\psi^\dagger \;=\; \mathcal{B}\,, \qquad \psi(\partial_2^i - \zeta\partial_1^i)\psi^\dagger \;=\; \mathcal{C}^i\,. \tag{4.1}$$

Recall that $\psi^\dagger := (\psi(x^a, \eta_i^\alpha, \bar{\zeta}))^\dagger$ and $(\mathcal{A}, \mathcal{B}, \mathcal{C}^i)$ depend only on x^a and η_i^α. The central idea is to demand analyticity in the spectral parameter ζ, which strongly restricts the possible form of ψ. One way to exploit this constraint starts from the observation that the left hand sides of (4.1) as well as of the reality condition (3.15) do not depend on ζ while ψ is expected to be a nontrivial function of ζ globally defined on $\mathbb{C}P^1$. Therefore, it must be a meromorphic function on $\mathbb{C}P^1$ possessing some poles which we choose to lie at finite points with constant coordinates $\mu_k \in \mathbb{C}P^1$.

Here we will build a (multi-soliton) solution ψ_m featuring m simple poles at positions $\mu_1, \ldots, \mu_m$ with[9] $\mathrm{Im}\,\mu_k < 0$ by left-multiplying an $(m-1)$-pole solution ψ_{m-1} with a single-pole factor of the form

$$\mathbb{1} \;+\; \frac{\mu_m - \bar{\mu}_m}{\zeta - \mu_m}\; P_m(x^a, \eta_i^\alpha)\,, \tag{4.2}$$

[9]This condition singles out solitons over anti-solitons, which appear for $\mathrm{Im}\,\mu_k > 0$.

where the $n\times n$ matrix function P_m is yet to be determined. Starting from the trivial (vacuum) solution $\psi_0 = \mathbb{1}$, the iteration $\psi_0 \mapsto \psi_1 \mapsto \ldots \mapsto \psi_m$ yields a multiplicative ansatz for ψ_m,

$$\psi_m \;=\; \prod_{\ell=0}^{m-1} \left(\mathbb{1} \;+\; \frac{\mu_{m-\ell} - \bar{\mu}_{m-\ell}}{\zeta - \mu_{m-\ell}}\; P_{m-\ell} \right), \tag{4.3}$$

which, via partial fraction decomposition, may be rewritten in the additive form

$$\psi_m \;=\; \mathbb{1} \;+\; \sum_{k=1}^{m} \frac{\Lambda_{mk} S_k^\dagger}{\zeta - \mu_k}\,, \tag{4.4}$$

where Λ_{mk} and S_k are some $n\times r_k$ matrices depending on x^a and η_i^α, with $r_k \leq n$.

Equations for S_k. Let us first consider the additive parametrization (4.4) of ψ_m. This ansatz must satisfy the reality condition (3.15) as well as our linear equations in the form (4.1). In particular, the poles at $\zeta = \bar{\mu}_k$ on the left hand sides of these equations have to be removable since the right hand sides are independent of ζ. Inserting the ansatz (4.4) and putting to zero the corresponding residues, we learn from (3.15) that

$$\left(\mathbb{1} \;+\; \sum_{\ell=1}^{m} \frac{\Lambda_{m\ell} S_\ell^\dagger}{\bar{\mu}_k - \mu_\ell} \right) S_k \;=\; 0\,, \tag{4.5}$$

while from (4.1) we obtain the differential equations

$$\left(\mathbb{1} \;+\; \sum_{\ell=1}^{m} \frac{\Lambda_{m\ell} S_\ell^\dagger}{\bar{\mu}_k - \mu_\ell} \right) \bar{L}_k^{\mathcal{A},\mathcal{B},i}\, S_k \;=\; 0\,, \tag{4.6}$$

where $\bar{L}_k^{\mathcal{A},\mathcal{B},i}$ stands for either

$$\bar{L}_k^{\mathcal{A}} \;=\; \partial_u - \bar{\mu}_k \partial_x\,, \qquad \bar{L}_k^{\mathcal{B}} \;=\; \mu_k(\partial_x - \bar{\mu}_k \partial_v) \qquad \text{or} \qquad \bar{L}_k^{i} \;=\; \partial_2^i - \bar{\mu}_k \partial_1^i\,. \tag{4.7}$$

Note that we consider a *recursive* procedure starting from m=1, and operators (4.7) will appear with $k = 1, \ldots, m$ if we consider poles at $\zeta = \bar{\mu}_k$.

Because the $\bar{L}_k^{\mathcal{A},\mathcal{B},i}$ for $k = 1, \ldots, m$ are linear differential operators, it is easy to write down the general solution for (4.6) at any given k, by passing from the coordinates $(u, v, x; \eta_i^1, \eta_i^2)$ to "co-moving coordinates" $(w_k, \bar{w}_k, s_k; \eta_k^i, \bar{\eta}_k^i)$. The precise relation for $k = 1, \ldots, m$ is [12, 58]

$$w_k \;:=\; x{+}\bar{\mu}_k u{+}\bar{\mu}_k^{-1} v \;=\; x{+}\tfrac{1}{2}(\bar{\mu}_k{-}\bar{\mu}_k^{-1})y{+}\tfrac{1}{2}(\bar{\mu}_k{+}\bar{\mu}_k^{-1})t \qquad \text{and} \qquad \eta_k^i \;:=\; \eta_i^1{+}\bar{\mu}_k \eta_i^2\,, \tag{4.8}$$

with $\bar{w}_k$ and $\bar{\eta}_k^i$ obtained by complex conjugation and the co-moving time s_k being inessential because by definition nothing will depend on it. The kth moving frame travels with a constant velocity

$$(\mathrm{v}_x\,,\,\mathrm{v}_y)_k \;=\; -\left(\frac{\mu_k + \bar{\mu}_k}{\mu_k\bar{\mu}_k + 1}\,,\, \frac{\mu_k\bar{\mu}_k - 1}{\mu_k\bar{\mu}_k + 1} \right), \tag{4.9}$$

so that the static case $w_k{=}z$ is recovered for $\mu_k = -\mathrm{i}$. On functions of $(w_k, \eta^i_k, \bar{w}_k, \bar{\eta}^i_k)$ alone the operators (4.7) act as

$$\bar{L}^{\mathcal{A}}_k \;=\; \bar{L}^{\mathcal{B}}_k \;=\; (\mu_k{-}\bar{\mu}_k)\frac{\partial}{\partial\bar{w}_k} \;=:\; \bar{L}_k \qquad\text{and}\qquad \bar{L}^i_k \;=\; (\mu_k{-}\bar{\mu}_k)\frac{\partial}{\partial\bar{\eta}^i_k}\;. \tag{4.10}$$

By induction in $k = 1,\dots,m$ we learn that, due to (4.5), a necessary and sufficient condition for a solution of (4.6) is

$$\bar{L}_k S_k \;=\; S_k \tilde{Z}_k \qquad\text{and}\qquad \bar{L}^i_k S_k \;=\; S_k \tilde{Z}^i_k \tag{4.11}$$

with some $r_k{\times}r_k$ matrices $\tilde{Z}_k$ and $\tilde{Z}^i_k$ depending on $(w_k, \bar{w}_k, \eta^j_k, \bar{\eta}^j_k)$.

Passing to the noncommutative bosonic coordinates we obtain

$$\big[\hat{w}_k\,,\,\hat{\bar{w}}_k\big] \;=\; 2\theta\,\nu_k\bar{\nu}_k \qquad\text{with}\qquad \nu_k\bar{\nu}_k \;=\; \frac{4\mathrm{i}}{\mu_k-\bar{\mu}_k-\mu_k^{-1}+\bar{\mu}_k^{-1}}\;. \tag{4.12}$$

Thus, we can introduce annihilation and creation operators

$$c_k \;=\; \frac{1}{\sqrt{2\theta}}\frac{\hat{w}_k}{\nu_k} \qquad\text{and}\qquad c^\dagger_k \;=\; \frac{1}{\sqrt{2\theta}}\frac{\hat{\bar{w}}_k}{\bar{\nu}_k} \qquad\text{so that}\quad [c_k\,,\,c^\dagger_k] \;=\; 1 \tag{4.13}$$

for $k = 1,\dots,m$. Naturally, this Heisenberg algebra is realized on a "co-moving" Fock space $\mathcal{H}_k$, with basis states $|\ell\rangle_k$ and a "co-moving" vacuum $|0\rangle_k$ subject to $c_k|0\rangle_k = 0$. Each co-moving vacuum $|0\rangle_k$ (annihilated by c_k) is related to the static vacuum $|0\rangle$ (annihilated by a) through an ISU(1,1) squeezing transformation (cf. [12]) which is time-dependent. The fermionic coordinates η^i_k and $\bar{\eta}^i_k$ remain spectators in the deformation. Coordinate derivatives are represented in the standard fashion as

$$\nu_k\sqrt{2\theta}\frac{\partial}{\partial w_k} \quad\mapsto\quad -[c^\dagger_k\,,\,\cdot\,] \qquad\text{and}\qquad \bar{\nu}_k\sqrt{2\theta}\frac{\partial}{\partial\bar{w}_k} \quad\mapsto\quad [c_k\,,\,\cdot\,]\;. \tag{4.14}$$

After the Moyal deformation, the $n{\times}r_k$ matrices S_k have become operator-valued, but are still functions of the Grassmann coordinates η^i_k and $\bar{\eta}^i_k$. The noncommutative version of the BPS conditions (4.11) naturally reads

$$c_k\,S_k \;=\; S_k\,Z_k \qquad\text{and}\qquad \frac{\partial}{\partial\bar{\eta}^i_k}\,S_k \;=\; S_k\,Z^i_k \tag{4.15}$$

where Z_k and Z^i_k are some operator-valued $r_k{\times}r_k$ matrix functions of η^j_k and $\bar{\eta}^j_k$.

Nonabelian solutions for S_k. For general data Z_k and Z^i_k it is difficult to solve (4.15), but it is also unnecessary because the final expression ψ_m turns out not to depend on them. Therefore, we conveniently choose

$$Z_k = c_k\otimes\mathbb{1}_{r_k\times r_k} \quad\text{and}\quad Z^i_k = 0 \qquad\Rightarrow\qquad S_k \;=\; R_k(c_k,\eta^i_k)\,, \tag{4.16}$$

where R_k is an arbitrary $n{\times}r_k$ matrix function independent of $c^\dagger_k$ and $\bar{\eta}^i_k$.[10] It is known that nonabelian (multi-) solitons arise for algebraic functions R_k (cf. e.g. [7] for the commutative and [12] for the noncommutative $\mathcal{N}{=}0$ case). Their common feature is a smooth

[10]Changing Z_k or Z^i_k multiplies R_k by an invertible factor from the right, which drops out later, except for the degenerate case $Z_k{=}0$ which yields $S_k = R_k\,|0\rangle_k\langle 0|_k$.

commutative limit. The only novelty of the supersymmetric extension is the η_k^i dependence, i.e.

$$R_k \;=\; R_{k,0} + \eta_k^i R_{k,i} + \eta_k^i \eta_k^j R_{k,ij} + \eta_k^i \eta_k^j \eta_k^p R_{k,ijp} + \eta_k^i \eta_k^j \eta_k^p \eta_k^q R_{k,ijpq} \;. \tag{4.17}$$

Abelian solutions for S_k. It is useful to view S_k as a map from $\mathbb{C}^{r_k} \otimes \mathcal{H}_k$ to $\mathbb{C}^n \otimes \mathcal{H}_k$ (momentarily suppressing the η dependence). The noncommutative setup now allows us to generalize the domain of this map to *any* subspace of $\mathbb{C}^n \otimes \mathcal{H}_k$. In particular, we may choose it to be finite-dimensional, say $\mathbb{C}^{q_k}$, and represent the map by an $n{\times}q_k$ array $|S_k\rangle$ of kets in $\mathcal{H}$. In this situation, Z_k and Z_k^i in (4.15) are just *number*-valued $q_k{\times}q_k$ matrix functions of η_k^j and $\bar\eta_k^j$. In case they do not depend on $\bar\eta_k^j$, we can write down the most general solution as

$$|S_k\rangle \;=\; R_k(c_k, \eta_k^j)\; |Z_k\rangle\; \exp\bigl\{\textstyle\sum_i Z_k^i(\eta_k^j)\, \bar\eta_k^i\bigr\} \qquad \text{with} \quad |Z_k\rangle \;:=\; \exp\bigl\{Z_k(\eta_k^j)\, c_k^\dagger\bigr\}\, |0\rangle_k \;. \tag{4.18}$$

As before, we may put $Z_k^i = 0$ without loss of generality, but now the choice of Z_k does matter.

For any given k generically there exists a q_k-dimensional basis change which diagonalizes the ket-valued matrix

$$|Z_k\rangle \quad \mapsto \quad \mathrm{diag}\bigl(\mathrm{e}^{\alpha_k^1 c^\dagger}, \mathrm{e}^{\alpha_k^2 c^\dagger}, \ldots, \mathrm{e}^{\alpha_k^{q_k} c^\dagger}\bigr)\, |0\rangle_k \;=\; \mathrm{diag}\bigl(|\alpha_k^1\rangle\,,\, |\alpha_k^2\rangle\,, \ldots,\, |\alpha_k^{q_k}\rangle\bigr)\,, \tag{4.19}$$

where we defined coherent states

$$|\alpha_k^l\rangle \;:=\; \mathrm{e}^{\alpha_k^l c^\dagger} |0\rangle_k \qquad \text{so that} \qquad c_k\, |\alpha_k^l\rangle \;=\; \alpha_k^l\, |\alpha_k^l\rangle \qquad \text{for} \quad l = 1, \ldots, q_k \quad \text{and} \quad \alpha_k^l \in \mathbb{C}\;. \tag{4.20}$$

Note that not only the entries of R_k but also the α_k^l are holomorphic functions of the co-moving Grassmann parameters η_k^j and thus can be expanded like in (4.17). In the U(1) model, we must use ket-valued $1{\times}q_k$ matrices $|S_k\rangle$ for all k, yielding rows

$$|S_k\rangle \;=\; \bigl(R_k^1\, |\alpha_k^1\rangle\,,\, R_k^2\, |\alpha_k^2\rangle\,, \ldots,\, R_k^{q_k}\, |\alpha_k^{q_k}\rangle\bigr) \qquad \text{for} \quad k = 1, \ldots, m\,, \tag{4.21}$$

with functions $\alpha_k^l(\eta_k^j)$. Here, the R_k^l only affect the states' normalization and can be collected in a diagonal matrix to the right, hence will drop out later and thus may all be put to one. Formally, we have recovered the known abelian (multi-) soliton solutions, but the supersymmetric extension has generalized $|S_k\rangle \to |S_k(\eta_k^j)\rangle$.

Explicit form of P_k. Let us now consider the multiplicative parametrization (4.3) of ψ_m which also allows us to solve (4.5). First of all, note that the reality condition (3.15) is satisfied if

$$P_k \;=\; P_k^\dagger \;=\; P_k^2 \qquad \Leftrightarrow \qquad P_k \;=\; T_k\, (T_k^\dagger T_k)^{-1} T_k^\dagger \qquad \text{for} \quad k = 1, \ldots, m\,, \tag{4.22}$$

meaning that P_k is an operator-valued hermitian projector (of group-space rank $r_k \le n$) built from an $n{\times}r_k$ matrix function T_k (the abelian case of $n{=}1$ is included). The reality condition follows just because

$$\Bigl(\mathbb{1} \;+\; \frac{\mu_k - \bar\mu_k}{\zeta - \mu_k}\; P_k\Bigr)\Bigl(\mathbb{1} \;+\; \frac{\bar\mu_k - \mu_k}{\zeta - \bar\mu_k}\; P_k\Bigr) \;=\; \mathbb{1} \qquad \text{for any } \zeta \text{ and } k = 1, \ldots, m\;. \tag{4.23}$$

The r_k columns of T_k span the image of P_k and obey

$$P_k\, T_k \;=\; T_k \qquad \Leftrightarrow \qquad (\mathbb{1}{-}P_k)\, T_k \;=\; 0\ . \tag{4.24}$$

Furthermore, the equation (4.5) with $m = k$ (induction) rewritten in the form

$$(\mathbb{1}{-}P_k)\, \prod_{\ell=1}^{k-1} \Big(\mathbb{1} \;+\; \frac{\mu_{k-\ell} - \bar{\mu}_{k-\ell}}{\bar{\mu}_k - \mu_{k-\ell}}\; P_{k-\ell} \Big)\; S_k \;=\; 0 \tag{4.25}$$

reveals that (cf. (4.24))

$$T_1 \;=\; S_1 \qquad \text{and} \qquad T_k \;=\; \Big\{ \prod_{\ell=1}^{k-1} \Big(\mathbb{1} \;-\; \frac{\mu_{k-\ell} - \bar{\mu}_{k-\ell}}{\mu_{k-\ell} - \bar{\mu}_k}\; P_{k-\ell} \Big) \Big\}\; S_k \qquad \text{for} \quad k \geq 2\,, \tag{4.26}$$

where the explicit form of S_k for $k = 1, \ldots, m$ is given in (4.16) or (4.18). The final result reads

$$\psi_m \;=\; \prod_{\ell=0}^{m-1} \Big(\mathbb{1} \;+\; \frac{\mu_{m-\ell} - \bar{\mu}_{m-\ell}}{\zeta - \mu_{m-\ell}}\; P_{m-\ell} \Big) \;=\; \mathbb{1} \;+\; \sum_{k=1}^{m} \frac{\Lambda_{mk} S_k^\dagger}{\zeta - \mu_k} \tag{4.27}$$

with hermitian projectors P_k given by (4.22), T_k given by (4.26) and S_k given by (4.16) or (4.18). The explicit form of Λ_{mk} (which we do not need) can be found in [12]. The corresponding superfields Φ and Υ are

$$\Phi_m \;=\; \psi_m^\dagger|_{\zeta=0} \;=\; \prod_{k=1}^{m} (\mathbb{1} - \rho_k P_k) \qquad \text{with} \quad \rho_k \;=\; 1 - \frac{\mu_k}{\bar{\mu}_k}\,, \tag{4.28a}$$

$$\Upsilon_m \;=\; \lim_{\zeta\to\infty} \zeta\, (\psi_m - \mathbb{1}) \;=\; \sum_{k=1}^{m} (\mu_k{-}\bar{\mu}_k)\, P_k\ . \tag{4.28b}$$

From (4.22) it is obvious that P_k is invariant under a similarity transformation

$$T_k \;\mapsto\; T_k\, \Lambda_k \qquad \Leftrightarrow \qquad S_k \;\mapsto\; S_k\, \Lambda_k \tag{4.29}$$

for an invertible operator-valued $r_k{\times}r_k$ matrix Λ_k. This justifies putting $Z_k^i = 0$ from the beginning and also the restriction to $Z_k = c_k \otimes \mathbb{1}_{r_k\times r_k}$ in the nonabelian case, both without loss of generality. Hence, the nonabelian solution space constructed here is parametrized by the set $\{R_k\}_1^m$ of matrix-valued functions of c_k and η_k^i and the pole positions μ_k. The abelian moduli space, however, is larger by the set $\{Z_k\}_1^m$ of matrix-values functions of η_k^i which generically contain the coherent-state parameter functions $\{\alpha_k^l(\eta_k^i)\}$. Restricting to $\eta_k^i{=}0$ reproduces the soliton configurations of the bosonic model [12].

Static solutions. Let us consider the reduction to 2+0 dimensions, i.e. the static case. Recall that static solutions correspond to the choice $m = 1$ and $\mu_1 \equiv \mu = -\mathrm{i}$ implying $w_1 = z$, so we drop the index k. Specializing (4.27), we have

$$\psi \;=\; \mathbb{1} \;-\; \frac{2\,\mathrm{i}}{\zeta + \mathrm{i}}\, P \qquad \text{so that} \qquad \Phi \;=\; \Phi^\dagger \;=\; \mathbb{1} - 2P\,, \tag{4.30}$$

where a hermitian projector P of group-space rank r satisfies the BPS equations

$$(\mathbb{1}-P)\,a\,P \;=\; 0 \qquad \Rightarrow \qquad (\mathbb{1}-P)\,a\,T \;=\; 0\,, \tag{4.31a}$$

$$(\mathbb{1}-P)\,\frac{\partial}{\partial\bar{\eta}^i}P \;=\; 0 \qquad \Rightarrow \qquad (\mathbb{1}-P)\,\frac{\partial}{\partial\bar{\eta}^i}T \;=\; 0\,, \tag{4.31b}$$

with $P = T\,(T^\dagger T)^{-1}T^\dagger$ and $\eta^i = \eta^1_i + \mathrm{i}\eta^2_i$. In this case $T = S$, and for a nonabelian $r{=}1$ projector P we get $T = T(a,\eta^i)$ as an $n{\times}1$ column. For the simplest case of $\mathcal{N}{=}1$ we just have (cf. [59])

$$T \;=\; T_e(a) \;+\; \eta\,T_o(a) \qquad \text{with} \quad \eta = \eta^1 + \mathrm{i}\eta^2\,, \tag{4.32}$$

where $T_e(a)$ and $T_o(a)$ are rational functions of a (e.g. polynomials) taking values in the even and odd parts of the Grassmann algebra. Similarly, an abelian $\mathcal{N}{=}1$ projector (for $n{=}1$) is built from

$$|T\rangle \;=\; \big(|\alpha^1\rangle\,,\,|\alpha^2\rangle\,,\ldots,\,|\alpha^q\rangle\big)\;. \tag{4.33}$$

At $\theta{=}0$, the static solution (4.32) of our supersymmetric U(n) sigma model is also a solution of the standard $\mathcal{N}{=}1$ supersymmetric $\mathbb{C}P^{n-1}$ sigma model in two dimensions (see e.g. [59]).[11] For this reason, one can overcome the previously mentioned difficulty with constructing an action (or energy from the viewpoint of 2+1 dimensions) for static configurations. Moreover, on solutions obeying the BPS conditions (4.31) the topological charge

$$\mathcal{Q} \;=\; 2\pi\theta\int\!\mathrm{d}\eta^1\mathrm{d}\eta^2\;\mathrm{Tr}\;\mathrm{tr}\;\Phi\,\big\{D_+\Phi\,,D_-\Phi\big\} \tag{4.34}$$

is proportional to the action (BPS bound)

$$S \;=\; 2\pi\theta\int\!\mathrm{d}\eta^1\mathrm{d}\eta^2\;\mathrm{Tr}\;\mathrm{tr}\;\big[D_+\Phi\,,D_-\Phi\big] \tag{4.35}$$

and is finite for algebraic functions T_e and T_o. Here, the standard superderivatives $D_\pm$ are defined as

$$D_+ \;=\; \frac{\partial}{\partial\eta} + \mathrm{i}\eta\,\partial_z \qquad \text{and} \qquad D_- \;=\; \frac{\partial}{\partial\bar{\eta}} + \mathrm{i}\bar{\eta}\,\partial_{\bar{z}}\;. \tag{4.36}$$

One-soliton configuration. For one moving soliton, from (4.27) and (4.28) we obtain

$$\psi_1 \;=\; \mathbb{1} \;+\; \frac{\mu-\bar{\mu}}{\zeta-\mu}\,P \qquad \text{with} \qquad P \;=\; T\,(T^\dagger T)^{-1}T^\dagger \tag{4.37}$$

and

$$\Phi \;=\; \mathbb{1} \;-\; \rho\,P \qquad \text{with} \qquad \rho \;=\; 1-\frac{\mu}{\bar{\mu}}\;. \tag{4.38}$$

Now our $n{\times}r$ matrix T must satisfy (putting $Z^i = 0$ and $Z = c\otimes\mathbb{1}_{r\times r}$)

$$[c\,,\,T] \;=\; 0 \qquad \text{and} \qquad \frac{\partial}{\partial\bar{\eta}^i}\,T \;=\; 0 \qquad \text{with} \quad \eta^i = \eta^1_i + \bar{\mu}\,\eta^2_i\,, \tag{4.39}$$

where c is the moving-frame annihilation operator given by (4.13) for $k{=}1$.

[11] In fact, Φ in (4.30) takes values in the Grassmannian $\mathrm{Gr}(r,n)$, and $\mathrm{Gr}(1,n) = \mathbb{C}P^{n-1}$.

JHEP06(2007)065

Recall that the operators c and $c^\dagger$ and therefore the matrix T and the projector P can be expressed in terms of the corresponding static objects by a unitary squeezing transformation (see e.g. (4.8) and (4.13)). For simplicity we again consider the case $\mathcal{N}$=1 and a nonabelian projector with r=1. Then (4.39) tells us that T is a holomorphic function of c and η, i.e.

$$T \;=\; T_e(c) \;+\; \eta\, T_o(c) \;=\; \begin{pmatrix} T_e^1(c) \;+\; \eta\, T_o^1(c) \\ \vdots \\ T_e^n(c) \;+\; \eta\, T_o^n(c) \end{pmatrix} \tag{4.40}$$

with polynomials T_e^a and T_o^a of order q, say, analogously to the static case (4.32). Note that, for T_o^a to be Grassmann-odd and nonzero, some extraneous Grassmann parameter must appear. Similarly, abelian projectors for a moving one-soliton obtain by subjecting (4.33) to a squeezing transformation.

For $\mathcal{N}$=1 the moving frame was defined in (4.8) (dropping the index k) via

$$w \;=\; x \;+\; \frac{1}{2}(\bar{\mu}-\bar{\mu}^{-1})y \;+\; \frac{1}{2}(\bar{\mu}+\bar{\mu}^{-1})t \qquad \text{and} \qquad \eta \;=\; \eta^1 + \bar{\mu}\eta^2 \qquad \text{hence} \quad \partial_t \eta = 0 \;. \tag{4.41}$$

Consider the moving frame with the coordinates $(w, \bar{w}, s; \eta, \bar{\eta})$ with the choice $s = t$ and the related change of the derivatives (see [12, 58])

$$\partial_x \;=\; \partial_w \;+\; \partial_{\bar{w}}\,, \tag{4.42a}$$
$$\partial_y \;=\; \frac{1}{2}(\bar{\mu}-\bar{\mu}^{-1})\,\partial_w \;+\; \frac{1}{2}(\mu-\mu^{-1})\,\partial_{\bar{w}}\,, \tag{4.42b}$$
$$\partial_t \;=\; \frac{1}{2}(\bar{\mu}+\bar{\mu}^{-1})\,\partial_w \;+\; \frac{1}{2}(\mu+\mu^{-1})\,\partial_{\bar{w}} \;+\; \partial_s\,, \tag{4.42c}$$
$$\partial_{\eta^1} \;=\; \partial_\eta \;+\; \partial_{\bar{\eta}}\,, \tag{4.42d}$$
$$\partial_{\eta^2} \;=\; \bar{\mu}\,\partial_\eta \;+\; \mu\,\partial_{\bar{\eta}}\;. \tag{4.42e}$$

In the moving frame our solution (4.38) is static, i.e. $\partial_s \Phi = 0$, and the projector P has the same form as in the static case. The only difference is the coefficient ρ instead of 2 in (4.38). Therefore, by computing the action (4.35) in $(w, \bar{w}; \eta^1, \eta^2)$ coordinates, we obtain for algebraic functions T in (4.40) a finite answer, which differs from the static one by a kinematical prefactor depending on μ (cf. [12] for the bosonic case).

Large-time asymptotics. Note that in the distinguished $(z, \bar{z}, t)$ coordinate frame (4.41) implies that at large times $w \to \kappa\, t$ with $\kappa = \frac{1}{2}(\bar{\mu}+\bar{\mu}^{-1})$. As a consequence, the t^q term in each polynomial in (4.40) will dominate, i.e.

$$T \quad\to\quad t^q \begin{pmatrix} a_1 \;+\; \eta\, b_1 \\ \vdots \\ a_n \;+\; \eta\, b_n \end{pmatrix} \;=:\; t^q\, \Gamma\,, \tag{4.43}$$

where Γ is a fixed vector in $\mathbb{C}^n$. It is easy to see that in the distinguished frame the large-time limit of Φ given by (4.38) is

$$\lim_{t\to\pm\infty} \Phi \;=\; \mathbb{1} \;-\; \rho\, \Pi \qquad \text{with} \qquad \Pi \;=\; \Gamma\,(\Gamma^\dagger\Gamma)^{-1}\Gamma^\dagger \tag{4.44}$$

being the projector on the constant vector Γ.

Consider now the m-soliton configuration (4.28). By induction of the above argument one easily arrives at the m-soliton generalization of (4.44). Namely, in the frame moving with the ℓth lump we have

$$\lim_{t\to\pm\infty}\Phi_m \;=\; (\mathbb{1}-\rho_1\Pi_1)\ldots(\mathbb{1}-\rho_{\ell-1}\Pi_{\ell-1})(\mathbb{1}-\rho_\ell P_\ell)(\mathbb{1}-\rho_{\ell+1}\Pi_{\ell+1})\ldots(\mathbb{1}-\rho_m\Pi_m)\,, \quad (4.45)$$

where the Π_m are constant projectors. This large-time factorization of multi-soliton solutions provides a proof of the no-scattering property because the asymptotic configurations are identical for large negative and large positive times.

5. Conclusions

In this paper we introduced a generalization of the modified integrable $\mathrm{U}(n)$ chiral model with $2\mathcal{N}\leq 8$ supersymmetries in 2+1 dimensions and considered a Moyal deformation of this model. It was shown that this $\mathcal{N}$-extended chiral model is equivalent to a gauge-fixed BPS subsector of an $\mathcal{N}$-extended super Yang-Mills model in 2+1 dimensions originating from twistor string theory. The dressing method was applied to generate a wide class of multi-soliton configurations, which are time-dependent finite-energy solutions to the equations of motion. Compared to the $\mathcal{N}{=}0$ model, the supersymmetric extension was seen to promote the configurations' building blocks to holomorphic functions of suitable Grassmann coordinates. By considering the large-time asymptotic factorization into a product of single soliton solutions we have shown that no scattering occurs within the dressing ansatz chosen here.

The considered model does not stand alone but is motivated by twistor string theory [37] with a target space reduced to the mini-supertwistor space [44, 45, 47]. In this context, the obtained multi-soliton solutions are to be regarded as $\mathrm{D}(0|2\mathcal{N})$-branes moving inside $\mathrm{D}(2|2\mathcal{N})$-branes [60]. Here $2\mathcal{N}$ appears due to fermionic worldvolume directions of our branes in the superspace description [60]. Switching on a constant B-field simply deforms the sigma model and D-brane worldvolumes noncommutatively, thereby admitting also regular supersymmetric noncommutative abelian solutions.

Restricting to static configurations, the models can be specialized to Grassmannian supersymmetric sigma models, where the superfield Φ takes values in $\mathrm{Gr}(r,n)$, and the field equations are invariant under $2\mathcal{N}$ supersymmetry transformations with $0\leq\mathcal{N}\leq 4$. This differs from the results for standard 2D sigma models [52, 53] where the target spaces have to be Kähler or hyper-Kähler for admitting two or four supersymmetries, respectively. This difference will be discussed in more details elsewhere.

We derived the supersymmetric chiral model in 2+1 dimensions through dimensional reduction and gauge fixing of the $\mathcal{N}$-extended supersymmetric SDYM equations in 2+2 dimensions. Recall that for the purely bosonic case most (if not all) integrable equations in three and fewer dimensions can be obtained from the SDYM equations (or their hierarchy [25]) by suitable dimensional reductions (see e.g. [61]–[65] and references therein). Moreover, this Ward conjecture [61] was extended to the noncommutative case (see e.g. [66, 67]). It will be interesting to consider similar reductions of the $\mathcal{N}$-extended supersymmetric SDYM equations (and their hierarchy [68]) to supersymmetric integrable equations in three and two dimensions generalizing earlier results [69].

Acknowledgments

We acknowledge fruitful discussions with C. Gutschwager. This work was supported in part by the Deutsche Forschungsgemeinschaft (DFG).

References

[1] M.R. Douglas and C.M. Hull, *D-branes and the noncommutative torus*, *JHEP* **02** (1998) 008 [hep-th/9711165].

[2] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219].

[3] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030 [hep-th/9903205].

[4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[5] O. Lechtenfeld, A.D. Popov and B. Spendig, *Open $N = 2$ strings in a B-field background and noncommutative self-dual Yang-Mills*, *Phys. Lett.* **B 507** (2001) 317 [hep-th/0012200].

[6] O. Lechtenfeld, A.D. Popov and B. Spendig, *Noncommutative solitons in open $N = 2$ string theory*, *JHEP* **06** (2001) 011 [hep-th/0103196].

[7] R.S. Ward, *Soliton solutions in an integrable chiral model in $2 + 1$ dimensions*, *J. Math. Phys.* **29** (1988) 386; *Classical solutions of the chiral model, unitons and holomorphic vector bundles*, *Commun. Math. Phys.* **128** (1990) 319.

[8] H. Ooguri and C. Vafa, *Self-duality and $N = 2$ string magic*, *Mod. Phys. Lett.* **A 5** (1990) 1389; *Geometry of $N = 2$ strings*, *Nucl. Phys.* **B 361** (1991) 469.

[9] M. Hamanaka, *Noncommutative solitons and D-branes*, hep-th/0303256;
F.A. Schaposnik, *Noncommutative solitons and instantons*, *Braz. J. Phys.* **34** (2004) 1349 [hep-th/0310202];
L. Tamassia, *Noncommutative supersymmetric/integrable models and string theory*, hep-th/0506064.

[10] M. Hamanaka, *Noncommutative solitons and integrable systems*, hep-th/0504001; *Noncommutative Ward's conjecture and integrable systems*, *Nucl. Phys.* **B 741** (2006) 368 [hep-th/0601209].

[11] W.J. Zakrzewski, *Low dimensional sigma models*, IOP Publishing, Bristol (1989).

[12] O. Lechtenfeld and A.D. Popov, *Noncommutative multi-solitons in $2 + 1$ dimensions*, *JHEP* **11** (2001) 040 [hep-th/0106213].

[13] O. Lechtenfeld and A.D. Popov, *Scattering of noncommutative solitons in $2 + 1$ dimensions*, *Phys. Lett.* **B 523** (2001) 178 [hep-th/0108118].

[14] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *JHEP* **05** (2000) 020 [hep-th/0003160].

[15] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons*, *Commun. Math. Phys.* **233** (2003) 355 [hep-th/0103256].

[16] M. Klawunn, O. Lechtenfeld and S. Petersen, *Moduli-space dynamics of noncommutative abelian sigma-model solitons*, *JHEP* **06** (2006) 028 [hep-th/0604219].

[17] S. Bieling, *Interaction of noncommutative plane waves in* $2+1$ *dimensions*, *J. Phys.* **A 35** (2002) 6281 [hep-th/0203269].

[18] M. Wolf, *Soliton antisoliton scattering configurations in a noncommutative sigma model in* $2+1$ *dimensions*, *JHEP* **06** (2002) 055 [hep-th/0204185].

[19] M. Ihl and S. Uhlmann, *Noncommutative extended waves and soliton-like configurations in* $N=2$ *string theory*, *Int. J. Mod. Phys.* **A 18** (2003) 4889 [hep-th/0211263].

[20] O. Lechtenfeld, L. Mazzanti, S. Penati, A.D. Popov and L. Tamassia, *Integrable noncommutative sine-Gordon model*, *Nucl. Phys.* **B 705** (2005) 477 [hep-th/0406065].

[21] A.V. Domrin, O. Lechtenfeld and S. Petersen, *Sigma-model solitons in the noncommutative plane: construction and stability analysis*, *JHEP* **03** (2005) 045 [hep-th/0412001].

[22] N. Berkovits, *SuperPoincaré invariant superstring field theory*, *Nucl. Phys.* **B 450** (1995) 90 [*Erratum ibid.* **459** (1996) 439] [hep-th/9503099].

[23] N. Berkovits and C. Vafa, $N=4$ *topological strings*, *Nucl. Phys.* **B 433** (1995) 123 [hep-th/9407190];
H. Ooguri and C. Vafa, *All loop* $N=2$ *string amplitudes*, *Nucl. Phys.* **B 451** (1995) 121 [hep-th/9505183].

[24] R.S. Ward and R.O. Wells, *Twistor geometry and field theory*, Cambridge University Press, Cambridge (1990).

[25] L.J. Mason and N.M.J. Woodhouse, *Integrability, self-duality, and twistor theory*, Oxford University Press, Oxford (1996).

[26] R.S. Ward, *On self-dual gauge fields*, *Phys. Lett.* **A 61** (1977) 81.

[27] O. Lechtenfeld and A.D. Popov, *On the integrability of covariant field theory for open* $N=2$ *strings*, *Phys. Lett.* **B 494** (2000) 148 [hep-th/0009144];
O. Lechtenfeld, A.D. Popov and S. Uhlmann, *Exact solutions of Berkovits' string field theory*, *Nucl. Phys.* **B 637** (2002) 119 [hep-th/0204155].

[28] A. Kling, O. Lechtenfeld, A.D. Popov and S. Uhlmann, *On nonperturbative solutions of superstring field theory*, *Phys. Lett.* **B 551** (2003) 193 [hep-th/0209186]; *Solving string field equations: new uses for old tools*, *Fortschr. Phys.* **51** (2003) 775 [hep-th/0212335];
A. Kling and S. Uhlmann, *String field theory vertices for fermions of integral weight*, *JHEP* **07** (2003) 061 [hep-th/0306254];
M. Ihl, A. Kling and S. Uhlmann, *String field theory projectors for fermions of integral weight*, *JHEP* **03** (2004) 002 [hep-th/0312314];
S. Uhlmann, *A note on kappa-diagonal surface states*, *JHEP* **11** (2004) 003 [hep-th/0408245].

[29] L.-L. Chau, M.-L. Ge and Y.-S. Wu, *Noether currents and algebraic structure of the hidden symmetry for superchiral fields*, *Phys. Rev.* **D 25** (1982) 1080;
L. Dolan, *A new symmetry group of real self-dual Yang-Mills*, *Phys. Lett.* **B 113** (1982) 387;

L.-L. Chau, M.-L. Ge, A. Sinha and Y.-S. Wu, *Hidden symmetry algebra for the self-dual Yang-Mills equation*, *Phys. Lett.* **B 121** (1983) 391;
L. Crane, *Action of the loop group on the self-dual Yang-Mills equation*, *Commun. Math. Phys.* **110** (1987) 391.

[30] A.D. Popov and C.R. Preitschopf, *Extended conformal symmetries of the self-dual Yang-Mills equations*, *Phys. Lett.* **B 374** (1996) 71 [hep-th/9512130];
T.A. Ivanova, *On current algebra of symmetries of the self-dual Yang-Mills equations*, *J. Math. Phys.* **39** (1998) 79 [hep-th/9702144];
A.D. Popov, *Self-dual Yang-Mills: symmetries and moduli space*, *Rev. Math. Phys.* **11** (1999) 1091 [hep-th/9803183]; *Holomorphic Chern-Simons-Witten theory: from 2D to 4D conformal field theories*, *Nucl. Phys.* **B 550** (1999) 585 [hep-th/9806239].

[31] T.A. Ivanova and O. Lechtenfeld, *Hidden symmetries of the open $N = 2$ string*, *Int. J. Mod. Phys.* **A 16** (2001) 303 [hep-th/0007049].

[32] A.M. Semikhatov, *Supersymmetric instanton*, *Phys. Lett.* **B 120** (1983) 171;
I.V. Volovich, *Super-self-duality for supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 123** (1983) 329.

[33] W. Siegel, *The $N = 2, N = 4$ string is self-dual $N = 4$ Yang-Mills*, *Phys. Rev.* **D 46** (1992) 3235 [hep-th/9205075].

[34] W. Siegel, *The $N = 4$ string is the same as the $N = 2$ string*, *Phys. Rev. Lett.* **69** (1992) 1493 [hep-th/9204005]; *Green-Schwarz formulation of self-dual superstring*, *Phys. Rev.* **D 47** (1993) 2512 [hep-th/9210008].

[35] N. Berkovits and W. Siegel, *Covariant field theory for self-dual strings*, *Nucl. Phys.* **B 505** (1997) 139 [hep-th/9703154].

[36] S. Bellucci, A. Galajinsky and O. Lechtenfeld, *A heterotic $N = 2$ string with space-time supersymmetry*, *Nucl. Phys.* **B 609** (2001) 410 [hep-th/0103049].

[37] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[38] N. Berkovits, *An alternative string theory in twistor space for $N = 4$ super-Yang-Mills*, *Phys. Rev. Lett.* **93** (2004) 011601 [hep-th/0402045];
N. Berkovits and L. Motl, *Cubic twistorial string field theory*, *JHEP* **04** (2004) 056 [hep-th/0403187].

[39] W. Siegel, *Untwisting the twistor superstring*, hep-th/0404255;
O. Lechtenfeld and A.D. Popov, *Supertwistors and cubic string field theory for open $N = 2$ strings*, *Phys. Lett.* **B 598** (2004) 113 [hep-th/0406179].

[40] I.A. Bandos, J.A. de Azcarraga and C. Miquel-Espanya, *Superspace formulations of the (super)twistor string*, *JHEP* **07** (2006) 005 [hep-th/0604037];
M. Abou-Zeid, C.M. Hull and L.J. Mason, *Einstein supergravity and new twistor string theories*, hep-th/0606272;
L. Dolan and P. Goddard, *Tree and loop amplitudes in open twistor string theory*, hep-th/0703054.

[41] A.D. Popov and C. Saemann, *On supertwistors, the Penrose-Ward transform and $N = 4$ super Yang-Mills theory*, *Adv. Theor. Math. Phys.* **9** (2005) 931 [hep-th/0405123];
A.D. Popov and M. Wolf, *Hidden symmetries and integrable hierarchy of the $N = 4$ supersymmetric Yang-Mills equations*, hep-th/0608225.

[42] C. Saemann, *Aspects of twistor geometry and supersymmetric field theories within superstring theory*, hep-th/0603098;
M. Wolf, *On supertwistor geometry and integrability in super gauge theory*, hep-th/0611013.

[43] A. Neitzke and C. Vafa, $N = 2$ *strings and the twistorial Calabi-Yau*, hep-th/0402128.

[44] D.-W. Chiou, O.J. Ganor, Y.P. Hong, B.S. Kim and I. Mitra, *Massless and massive three dimensional super Yang-Mills theory and mini-twistor string theory*, *Phys. Rev.* **D 71** (2005) 125016 [hep-th/0502076];
D.-W. Chiou, O.J. Ganor and B.S. Kim, *A deformation of twistor space and a chiral mass term in* $N = 4$ *super Yang-Mills theory*, *JHEP* **03** (2006) 027 [hep-th/0512242].

[45] A.D. Popov, C. Saemann and M. Wolf, *The topological B-model on a mini-supertwistor space and supersymmetric Bogomolny monopole equations*, *JHEP* **10** (2005) 058 [hep-th/0505161].

[46] C. Saemann, *On the mini-superambitwistor space and* $N = 8$ *super Yang-Mills theory*, hep-th/0508137; *The mini-superambitwistor space*, hep-th/0511251.

[47] A.D. Popov, *Sigma models with* $N = 8$ *supersymmetries in* $2+1$ *and* $1+1$ *dimensions*, *Phys. Lett.* **B 647** (2007) 509 [hep-th/0702106].

[48] C. Devchand and V. Ogievetsky, *Interacting fields of arbitrary spin and* $N > 4$ *supersymmetric self-dual Yang-Mills equations*, *Nucl. Phys.* **B 481** (1996) 188 [hep-th/9606027].

[49] N. Seiberg, *Notes on theories with 16 supercharges*, *Nucl. Phys.* **67** *(Proc. Suppl.)* (1998) 158 [hep-th/9705117].

[50] A. Konechny and A.S. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, *Phys. Rept.* **360** (2002) 353 [hep-th/0012145] [hep-th/0107251];
M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048];
R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[51] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, Berlin (1987).

[52] B. Zumino, *Supersymmetry and Kähler manifolds*, *Phys. Lett.* **B 87** (1979) 203.

[53] L. Alvarez-Gaumé and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, *Commun. Math. Phys.* **80** (1981) 443.

[54] V.P. Nair and J. Schiff, *Kahler Chern-Simons theory and symmetries of anti-self-dual gauge fields*, *Nucl. Phys.* **B 371** (1992) 329;
A. Losev, G.W. Moore, N. Nekrasov and S. Shatashvili, *Four-dimensional avatars of two-dimensional rCFT*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 130 [hep-th/9509151].

[55] E.T. Newman, *Source-free Yang-Mills theories*, *Phys. Rev.* **D 18** (1978) 2901;
A.N. Leznov, *On equivalence of four-dimensional self-duality equations to continual analog of the main chiral field problem.* (In Russian), *Teor. Mat. Fiz.* **73** (1988) 1233.

[56] T.A. Ioannidou and W. Zakrzewski, *Lagrangian formulation of the general modified chiral model*, *Phys. Lett.* **A 249** (1998) 303 [hep-th/9802177].

[57] V.E. Zakharov and A.V. Mikhailov, *Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique. (in Russian)*, *Sov. Phys. JETP* **47** (1978) 1017;
V.E. Zakharov and A.B. Shabat, *Integration of nonlinear equations by the inverse scattering method*, *Funct. Anal. Appl.* **13** (1979) 166;
P. Forgacs, Z. Horvath and L. Palla, *Solution generating technique for self-dual monopoles*, *Nucl. Phys.* **B 229** (1983) 77;
K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, *J. Diff. Geom.* **30** (1989) 1;
O. Babelon and D. Bernard, *Dressing symmetries*, *Commun. Math. Phys.* **149** (1992) 279 [hep-th/9111036].

[58] C.-S. Chu and O. Lechtenfeld, *Time-space noncommutative abelian solitons*, *Phys. Lett.* **B 625** (2005) 145 [hep-th/0507062].

[59] A.M. Perelomov, *Supersymmetric chiral models: geometrical aspects*, *Phys. Rept.* **174** (1989) 229.

[60] O. Lechtenfeld and C. Saemann, *Matrix models and D-branes in twistor string theory*, *JHEP* **03** (2006) 002 [hep-th/0511130].

[61] R.S. Ward, *Integrable and solvable systems, and relations among them*, *Phil. Trans. Roy. Soc. Lond. A* **315** (1985) 451; *Multidimensional integrable systems* in *Field theory, quantum gravity and strings*, H.J. De Vega, N. Sanchez eds. (1986).

[62] L.J. Mason and G.A.J. Sparling, *Nonlinear Schrödinger and Korteweg-de Vries are reductions of self-dual Yang-Mills*, *Phys. Lett.* **A 137** (1989) 29; *Twistor correspondences for the soliton hierarchies*, *J. Geom. Phys.* **8** (1992) 243;
L.J. Mason and M.A. Singer, *The twistor theory of equations of KdV type. 1*, *Commun. Math. Phys.* **166** (1994) 191.

[63] S. Chakravarty, M.J. Ablowitz and P.A. Clarkson, *Reductions of self-dual Yang-Mills fields and classical systems*, *Phys. Rev. Lett.* **65** (1990) 1085;
M.J. Ablowitz, S. Chakravarty and L.A. Takhtajan, *A self-dual Yang-Mills hierarchy and its reductions to integrable systems in* $(1+1)$*-dimensions and* $(2+1)$*-dimensions*, *Commun. Math. Phys.* **158** (1993) 289;
S. Chakravarty, S.L. Kent and E.T. Newman, *Some reductions of the self-dual Yang-Mills equations to integrable systems in* $(2+1)$*-dimensions*, *J. Math. Phys.* **36** (1995) 763;
M.J. Ablowitz, S. Chakravarty and R.G. Halburd, *Integrable systems and reductions of the self-dual Yang-Mills equations*, *J. Math. Phys.* **44** (2003) 3147.

[64] G.V. Dunne, R. Jackiw, S.-Y. Pi and C.A. Trugenberger, *Self-dual Chern-Simons solitons and two-dimensional nonlinear equations*, *Phys. Rev.* **D 43** (1991) 1332;
I. Bakas and D.A. Depireux, *The origin of gauge symmetries in integrable systems of KdV type*, *Int. J. Mod. Phys.* **A 7** (1992) 1767.

[65] T.A. Ivanova and A.D. Popov, *Soliton equations and self-dual gauge fields*, *Phys. Lett.* **A 170** (1992) 293; *Some new integrable equations from the self-dual Yang-Mills equations*, *Phys. Lett.* **A 205** (1995) 158 [hep-th/9508129]; *Self-dual Yang-Mills fields in* $D=4$ *and integrable systems in* $1 \leq D \leq 3$, *Theor. Math. Phys.* **102** (1995) 280;
M. Legaré and A.D. Popov, *Reductions of a Lax pair for self-duality equations of the Yang-Mills model*, *Sov. Phys. JETP* **59** (1994) 833; *Lax pairs of integrable equations in*

$1 \leq D \leq 3$ *dimensions as reductions of the Lax pair for the self-dual Yang-Mills equations*, *Phys. Lett.* **A 198** (1995) 195.

[66] M. Hamanaka and K. Toda, *Towards noncommutative integrable systems*, *Phys. Lett.* **A 316** (2003) 77 [hep-th/0211148]; *Noncommutative Burgers equation*, *J. Phys.* **A 36** (2003) 11981–11998 [hep-th/0301213];
M. Hamanaka, *On reductions of noncommutative anti-self-dual Yang-Mills equations*, *Phys. Lett.* **B 625** (2005) 324 [hep-th/0507112].

[67] A. Dimakis and F. Mueller-Hoissen, *Bicomplexes and integrable models*, *J. Phys.* **A 33** (200) 6579 [nlin.si/0006029];
M.T. Grisaru, L. Mazzanti, S. Penati and L. Tamassia, *Some properties of the integrable noncommutative sine-Gordon system*, *JHEP* **04** (2004) 057 [hep-th/0310214];
I. Cabrera-Carnero, *Abelian Toda field theories on the noncommutative plane*, *JHEP* **10** (2005) 071 [hep-th/0503147].

[68] M. Wolf, *On hidden symmetries of a super gauge theory and twistor string theory*, *JHEP* **02** (2005) 018 [hep-th/0412163]; *Twistors and aspects of integrability of self-dual SYM theory*, hep-th/0511230.

[69] S.J. Gates Jr. and H. Nishino, *Supersymmetric soluble systems embedded in supersymmetric self-dual Yang-Mills theory*, *Phys. Lett.* **B 299** (1993) 255 [hep-th/9210163];
A.K. Das and C.A.P. Galvao, *Self-duality and the supersymmetric KdV hierarchy*, *Mod. Phys. Lett.* **A 8** (1993) 1399 [hep-th/9211014];
H. Nishino and S. Rajpoot, *Noncommutative self-dual supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 572** (2003) 91 [hep-th/0306290].

JHEP06(2007)065